Math 650-600: Several Complex Variables

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Definitions from (Noah) Webster's dictionary

several
> a: more than one
> b: more than two but fewer than many

complex number
> a number of the form $a + b\sqrt{-1}$ where a and b are real numbers

variable
> a quantity that may assume any one of a set of values

Our variables are elements of n-dimensional complex Euclidean space $\mathbb{C}^n$, where $n \geq 2$.

Notation: $z = (z_1, \ldots, z_n)$, where each $z_j \in \mathbb{C}$.

Aspects of the subject

Function theory: do there exist (and can one construct) holomorphic functions with specified properties?

Algebraic geometry: the functions are polynomials.

Complex manifolds and complex varieties: spaces modeled locally on $\mathbb{C}^n$.

Sheaves and cohomology: tools for passing from local to global information.

Complex differential geometry: curvature.

Partial differential equations: existence and regularity of solutions of the inhomogeneous Cauchy-Riemann equations.
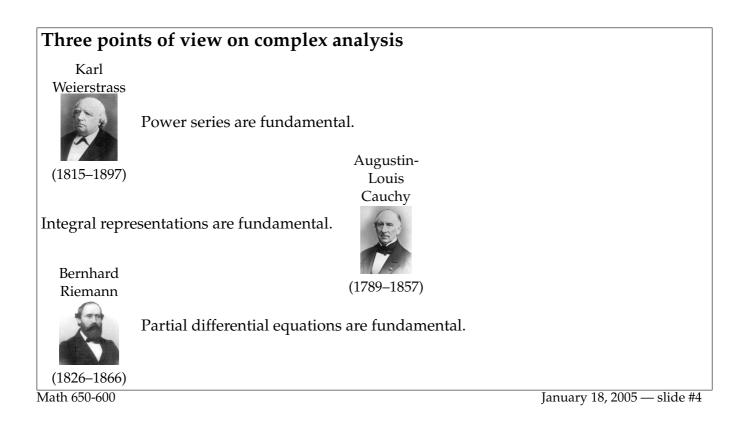
Harmonic analysis: pseudo-differential operators, Fourier analysis on groups.

Functional analysis: function spaces, Banach algebras.

Functions as mappings: classification of domains, automorphisms.

Three points of view on complex analysis

Karl
Weierstrass

Power series are fundamental.

Augustin-
Louis
Cauchy

(1815–1897)

Integral representations are fundamental.

(1789–1857)

Bernhard
Riemann

Partial differential equations are fundamental.

(1826–1866)

Why is one variable different from two?

1. Zeroes of non-trivial holomorphic functions of one variable are isolated, but zeroes of holomorphic functions of two or more variables are never isolated.
The same comment applies to singularities.

2. Convergence domains of one-variable power series are always discs (or the whole plane), but convergence domains of multi-variable power series can be more complicated.

3. For every boundary point of a domain in $\mathbb{C}$, there is a holomorphic function on the domain that blows up at that boundary point. In $\mathbb{C}^2$, however, *some* domains have boundary points at which every holomorphic function on the domain has a removable singularity.

$1 \neq 2$ **(continued)**

4. In $\mathbb{C}$, the Cauchy integral $\dfrac{1}{2\pi i} \displaystyle\oint \dfrac{f(\zeta)}{\zeta - z}\, d\zeta$, which reproduces holomorphic functions, has two special properties:

    a. The kernel $\frac{1}{\zeta - z}$ is holomorphic in the free variable.

    b. The kernel is universal (independent of the domain).

In higher dimensions, one can have a reproducing kernel with either property (a) or property (b), but not both at once.
The Bochner-Martinelli kernel is universal but not holomorphic in the free variable.
The Szegő kernel is holomorphic in the free variable but depends on the domain.

5. In $\mathbb{C}^2$, each domain has its own function theory. There is no Riemann mapping theorem in $\mathbb{C}^2$.

$1 \neq 2$ **(continued)**

6. In $\mathbb{C}$, the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

form a system of two equations in two unknown (real) functions.

In $\mathbb{C}^2$, the Cauchy-Riemann equations form a system of four equations in two unknown (real) functions: the system is *over-determined*.