Math 650-600: Several Complex Variables

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Reminders on one-variable power series

Theorem. If the power series $\sum_{n=0}^{\infty} c_n z^n$ converges when $z = w$, then the series converges absolutely when $|z| < |w|$ and uniformly on compact subsets of that disc.
Proof. Compare with a geometric series.

Corollary. The interior of the set on which the power series converges is a union of open discs centered at 0 and therefore is either an open disc or the whole plane.

Cauchy-Hadamard formula. The radius of the disc of convergence equals

$$\frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}.$$

Jacques Hadamard

(1865–1963)

Multi-variable power series: notation

Notation. In n dimensions, z means $(z_1, \ldots, z_n)$.

A multi-index α means an n-tuple of non-negative integers $(\alpha_1, \ldots, \alpha_n)$. Also $|\alpha|$ means the sum $\alpha_1 + \cdots + \alpha_n$, and $\alpha!$ means the product $\alpha_1! \cdots \alpha_n!$.

The monomial z^α means the product $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

A multi-variable power series has the form $\sum_\alpha c_\alpha z^\alpha$.

There is an ambiguity about the order in which the terms are added, so usually one considers absolute convergence (in which case the order of terms does not matter).

Multi-variable power series: first theorem

Theorem. If the power series $\sum_\alpha c_\alpha z^\alpha$ converges absolutely when $z = w$, then the series converges absolutely when $|z_j| < |w_j|$ for all j and uniformly on compact subsets of that polydisc.

Proof. By hypothesis, there is a constant M such that $|c_\alpha w^\alpha| \le M$ for all α.
Consider a compact polydisc such that $|z_j| \le \lambda |w_j|$ for all j, where $0 < \lambda < 1$.
In this compact polydisc, $|c_\alpha z^\alpha| \le |c_\alpha \lambda^{|\alpha|} w^\alpha| \le M\lambda^{|\alpha|}$.
Now $\sum_\alpha \lambda^{|\alpha|} = \sum_{\alpha_1=0}^\infty \cdots \sum_{\alpha_n=0}^\infty \lambda^{\alpha_1} \cdots \lambda^{\alpha_n} = \frac{1}{(1-\lambda)^n}$.
Therefore $\sum_\alpha c_\alpha z^\alpha$ converges uniformly and absolutely in the compact polydisc (by the Weierstrass M-test).

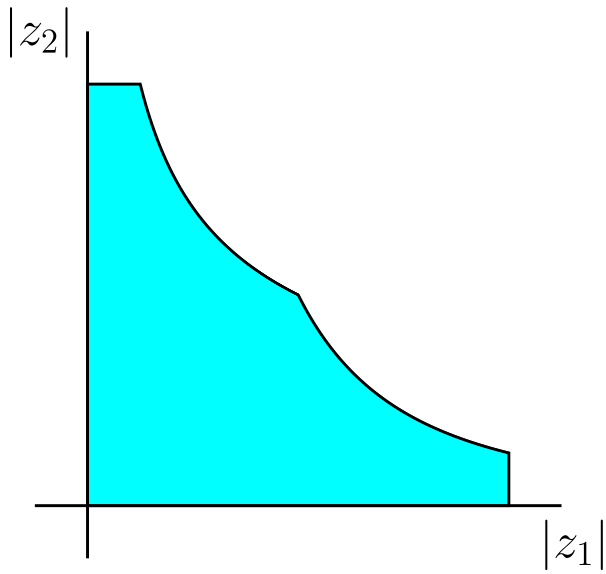
Interpretation of the theorem

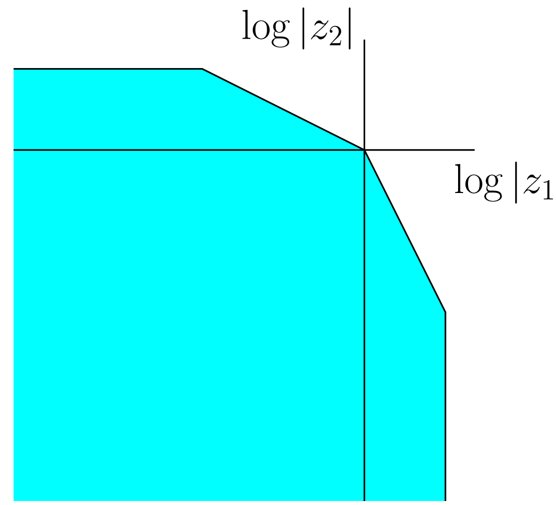
(a) The largest open set on which a multi-variable power series $\sum_\alpha c_\alpha z^\alpha$ converges absolutely is a union of open polydiscs centered at 0.

(b) The proof shows that this convergence domain also is the interior of the set of points z for which the terms $|c_\alpha z^\alpha|$ admit a bound that is independent of α.

Power series example

What is the convergence domain of the two-variable power series $\sum_{k=0}^{\infty}(\frac{1}{2^k}z_1^k + \frac{1}{2^k}z_2^k + z_1^k z_2^{2k} + z_1^{2k}z_2^k)$?





Properties of convergence domains

The following properties hold for the largest open set where a power series $\sum_\alpha c_\alpha z^\alpha$ converges absolutely.

(a) The domain is *multi-circular*: if a point $(z_1, \ldots, z_n)$ belongs to the domain, so does the point $(\lambda_1 z_1, \ldots, \lambda_n z_n)$ whenever $|\lambda_j| = 1$ for all j. Such a domain is also called a Reinhardt domain [after Karl August Reinhardt (1895–1941)].

(b) The multi-circular domain is *complete*: property (a) holds whenever $|\lambda_j| \leq 1$ for all j.

(c) The domain is *logarithmically convex*: the set of points x in $\mathbb{R}^n$ for which the point $(e^{x_1}, \ldots, e^{x_n})$ belongs to the domain is a convex set.