Math 650-600: Several Complex Variables

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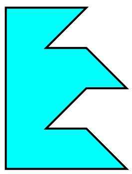
Karl August Reinhardt (1895–1941)

In 1928, Reinhardt solved the middle part of Hilbert's 18th problem: there exists a polyhedron that tiles three-dimensional space but that is not a fundamental domain for a group of rigid motions. Roughly speaking, this means that the symmetries of the tiling are not transitive.



In 1935, Heinrich Heesch found a planar example.

Heesch's tile

Review of convergence domains

The *convergence domain* of a multi-variable power series $\sum_\alpha c_\alpha z^\alpha$ means the largest open set on which the series converges absolutely.
The convergence is uniform on compact subsets of the convergence domain.

Theorem. The convergence domain of a power series is a complete Reinhardt domain that is logarithmically convex.

We already showed by comparing with geometric series that a convergence domain is a union of polydiscs centered at the origin, and this is just another way of saying that the domain is a complete Reinhardt domain.

Proof of logarithmic convexity

It suffices to prove logarithmic convexity of the set of points z for which $|c_\alpha z^\alpha|$ is bounded independently of α, because the convergence domain is the interior of that set.

If x and y are points of $\mathbb{R}^n$ in the logarithmic image, then there is a constant M such that

$$|c_\alpha e^{\alpha\cdot x}| \le M \qquad \text{and} \qquad |c_\alpha e^{\alpha\cdot y}| \le M \qquad \text{for all } \alpha,$$

where $\alpha \cdot x$ means $\alpha_1 x_1 + \cdots + \alpha_n x_n$.

Taking the geometric average gives

$$|c_\alpha e^{\alpha\cdot(tx+(1-t)y)}| \le M \qquad \text{when} \qquad 0 \le t \le 1.$$

Therefore the logarithmic image is indeed convex.

Preview of coming attraction

Theorem. A complete Reinhardt domain is a convergence domain for some power series if and only if the domain is logarithmically convex.

We have already established that a convergence domain is logarithmically convex.

The remaining problem is: given a complete logarithmically convex Reinhardt domain, construct a power series whose domain of convergence is the given domain.

Exercise. Find an explicit power series whose convergence domain is the unit ball in $\mathbb{C}^n$: namely, the set of points $(z_1, \ldots, z_n)$ for which $|z_1|^2 + \cdots + |z_n|^2 < 1$.

Definition of holomorphic function

Suppose that Ω is an open set in $\mathbb{C}^n$.
Theorem (and definition). If $f : \Omega \to \mathbb{C}$ has continuous first partial derivatives in the underlying real variables, then the following properties are equivalent (and this is easy to see).

A function satisfying these properties is called *holomorphic*.

1. In each open polydisc contained in Ω, the function f is represented by an absolutely convergent power series.
2. In each polydisc whose closure is contained in Ω, the function f is represented by the iterated Cauchy integral:

$$f(z_1, \dots, z_n) = \left(\frac{1}{2\pi i}\right)^n \oint \cdots \oint \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}\, d\zeta_1 \cdots d\zeta_n.$$

3. The function f satisfies the Cauchy-Riemann equations in each variable.

Some analogues of one-variable theory

• A holomorphic function has continuous partial derivatives of all orders.

• A holomorphic function in a polydisc (centered at 0) is represented by the power series $\sum_\alpha \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha$.

• **Identity theorem.** Two holomorphic functions on a connected open set agree identically as soon as they agree on some open subset.

• **Maximum principle.** The modulus of a non-constant holomorphic function on a connected open set cannot attain a maximum.

• **Exercise: Cauchy's estimates for derivatives.** If f is holomorphic in the polydisc $\{\, z \in \mathbb{C}^n : |z_j| < r_j \text{ for } 1 \le j \le n \,\}$, and if $|f|$ is bounded by a constant M, then $|f^{(\alpha)}(0)| \le M\alpha!/r^\alpha$.