Math 650-600: Several Complex Variables

Harold P. Boas
boas@tamu.edu

Exercises from last time

Are the following subsets of the unit bidisc D^2 domains of holomorphy? polynomially convex?

1. $D^2 \setminus \{ (x, 0) : 0 \leq x \}$
2. $D^2 \setminus \{ (x, 0) : -1 < x < 1 \}$
3. $D^2 \setminus \{ (z_1, 0) : 0 \leq \operatorname{Re} z_1 \}$
4. $D^2 \setminus \{ (z_1, 0) : |z_1| < 1 \}$

Reminders on subharmonic functions

A continuous (or upper semi-continuous) function u on a domain Ω in $\mathbb{C}$ is *subharmonic* if u satisfies any of the following equivalent properties.

1. Maximum principle: whenever h is a harmonic function on a closed disc in Ω, if $u \leq h$ on the boundary of the disc, then $u \leq h$ everywhere on the disc.
2. Sub-mean-value property:

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta$$

   whenever the closed disc $D(a, r)$ is contained in Ω.

3. Non-negative Laplacian: $\Delta u \geq 0$ in Ω.

(If u is not twice differentiable, the third property may be interpreted in the sense of distributions.)

Example: if f is holomorphic, then $\log |f|$ is subharmonic.

Plurisubharmonic functions

A continuous (or upper semi-continuous) function u on a domain Ω in $\mathbb{C}^n$ is *plurisubharmonic* if the restriction of u to every complex line is subharmonic: namely, for every point a in Ω and every direction b the function $\lambda \mapsto u(a + b\lambda)$ of the complex variable λ is subharmonic (where defined).

Example/Exercise. The function $\log(1 + |z_1|^2 + |z_2|^2)$ is plurisubharmonic in $\mathbb{C}^2$.

Property (3) for subharmonic functions implies that u is plurisubharmonic if and only if the $n \times n$ Hermitian matrix $\dfrac{\partial^2 u}{\partial z_j \partial \bar{z}_k}$ is positive semi-definite: $\displaystyle\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geq 0$ for all vectors t in $\mathbb{C}^n$.