Math 650-600: Several Complex Variables

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Plurisubharmonic functions: reminders

A continuous (or upper semi-continuous) function u on a domain Ω in $\mathbb{C}^n$ is *plurisubharmonic* if either of the following properties holds.

1. The restriction of u to every complex line is subharmonic: namely, for every point a in Ω and every direction b the function $\lambda \mapsto u(a+b\lambda)$ of the complex variable λ is subharmonic (where defined).

2. The $n \times n$ Hermitian matrix $\dfrac{\partial^2 u}{\partial z_j \partial \bar{z}_k}$ is positive semi-definite: $\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} t_j \bar{t}_k \geq 0$ for all vectors t in $\mathbb{C}^n$.
(If u is not class C^2, one can say that $i\partial\bar{\partial}u$ is a *positive current*.)

Pseudoconvexity

Theorem/definition. The following properties of a domain Ω in $\mathbb{C}^n$ are equivalent. A domain satisfying these properties is called *pseudoconvex*.

1. The domain Ω is convex with respect to the set of plurisubharmonic functions.
2. The function $z \mapsto -\log \operatorname{dist}(z, b\Omega)$ is plurisubharmonic.
3. There exists a continuous plurisubharmonic function u such that for every real number r, the sub-level set $\{\, z \in \Omega : u(z) < r \,\}$ is relatively compact in Ω.
   (Such a function is a *plurisubharmonic exhaustion function.*)

Example. If Ω is the unit ball, then $-\log \operatorname{dist}(z, b\Omega) = -\log(1-|z|)$, and this function is a plurisubharmonic exhaustion function.

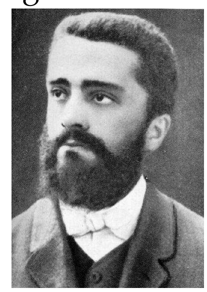
The Levi problem

Every domain of holomorphy is pseudoconvex because the hull with respect to the plurisubharmonic functions is contained in the hull with respect to the moduli of holomorphic functions.

E. E. Levi's problem.
Prove the converse: every pseudoconvex domain is a domain of holomorphy.

We will solve the problem later using $\bar{\partial}$ methods.

Eugenio Elia Levi



(1883–1917)

Proof of pseudoconvexity equivalents

1. Ω is convex with respect to plurisubharmonic functions.
2. $z \mapsto -\log \operatorname{dist}(z, b\Omega)$ is plurisubharmonic.
3. A continuous plurisubharmonic exhaustion function exists.

(2) $\Rightarrow$ (3).
$-\log \operatorname{dist}(z, b\Omega)$ is an exhaustion function if Ω is bounded.
In general, use $\max(|z|^2, -\log \operatorname{dist}(z, b\Omega))$.

(3) $\Rightarrow$ (1).
If $K \subset\subset \Omega$, then K is contained in some sublevel set of the plurisubharmonic exhaustion function. By definition of hull, the plurisubharmonic hull $\widehat{K}$ is contained in the same sublevel set. Therefore $\widehat{K} \subset\subset \Omega$.

It remains to prove that (1) $\Rightarrow$ (2).
The proof uses the *Kontinuitätssatz* of Hartogs.

Analytic discs

Definition. An *analytic disc* in $\mathbb{C}^n$ is the image of a map $(f_1, \ldots, f_n)$ of a closed disc in $\mathbb{C}$, where each component function f_j is holomorphic on the open disc and continuous on the closed disc. The *boundary* of an analytic disc means the image of the boundary of the one-dimensional disc.

Theorem. If Ω is convex with respect to the plurisubharmonic functions, then Ω satisfies the continuity principle (Kontinuitätssatz): namely, if $\{D_\alpha\}$ is a collection of analytic discs in Ω such that $\bigcup_\alpha bD_\alpha \subset\subset \Omega$, then $\bigcup_\alpha D_\alpha \subset\subset \Omega$.

Proof. By the maximum principle for subharmonic functions, each D_α is a subset of the plurisubharmonic hull of $bD\alpha$. The plurisubharmonic hull of $\bigcup_\alpha bD_\alpha$ is relatively compact by the hypothesis of plurisubharmonic convexity.

Exercises

1. Verify that if u is a plurisubharmonic function of class C^2 on a domain Ω, and if $F : G \to \Omega$ is a holomorphic mapping, then $u \circ F$ is a plurisubharmonic function on G.
The domains G and Ω may be in spaces of different dimensions.

2. By convolving with a mollifier, show that if u is an arbitrary plurisubharmonic function (not necessarily smooth), then u is the limit of a decreasing sequence of smooth plurisubharmonic functions.
More precisely, if u lives on Ω, then there is an exhaustion of Ω by nested subdomains Ω_j, and there is a smooth plurisubharmonic function u_j on Ω_j such that the sequence $u_j \downarrow u$.