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Lecture notes on several complex variables

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1 Introduction

Although Karl Weierstrass studied holomorphic functions of two variables already in the nineteenth century, the modern theory of several complex variables may be dated to the researches of Friedrich (Fritz) Hartogs (1874–1943) in the first decade of the twentieth century.[1]

Some parts of the theory of holomorphic functions—the maximum principle, for example—are essentially the same in all dimensions. The most interesting parts of the theory of several complex variables are the features that differ from the one-dimensional theory.

The one-dimensional theory is illuminated by several complementary points of view: power series, integral representations, partial differential equations, and geometry. The multi-dimensional theory reveals striking new phenomena from each of these points of view.

1.1 Power series

A one-variable power series converges inside a certain disc and diverges outside the closure of the disc. The convergence region for a two-dimensional power series, however, can have infinitely many different shapes. For instance, the largest open set in which the series $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} z^n w^m$ converges is the unit bidisc $\{\,(z,w) : |z| < 1 \text{ and } |w| < 1\,\}$, while the series $\sum_{n=0}^{\infty} z^n w^n$ converges in the unbounded hyperbolic region where $|zw| < 1$.

The theory of one-dimensional power series bifurcates into the theory of entire functions and the theory of functions on the unit disc. In higher dimensions, studying power series already leads to function theory on infinitely many different types of domains. A natural question, to be answered presently, is to characterize the domains that are convergence domains for multi-variable power series.

Exercise 1. Exhibit a two-variable power series whose convergence domain is the unit ball $\{\,(z,w) : |z|^2 + |w|^2 < 1\,\}$.

Hartogs discovered that a function holomorphic in a neighborhood of the boundary of the unit bidisc automatically extends to be holomorphic on the interior of the bidisc; one can prove this property by considering one-variable Laurent series on slices. Thus,

[1] A student of Pringsheim, Hartogs belonged to the Munich school of mathematicians. Because of their Jewish heritage, both Pringsheim and Hartogs suffered greatly under the Nazi regime in the 1930s. Pringsheim, a wealthy man, managed to buy his way out of Germany into Switzerland, where he died at an advanced age in 1941. The situation for Hartogs, however, grew ever more desperate, and in 1943 he chose to end his own life by an overdose of barbiturates rather than to be sent to a death camp.

in dramatic contrast to the situation in one variable, there are domains in $\mathbb{C}^2$ on which all holomorphic functions extend to a larger domain. A natural question, to be answered presently, is to characterize the *domains of holomorphy*, that is, the natural domains of existence of holomorphic functions.

The discovery of Hartogs also shows that holomorphic functions of several variables never have isolated singularities and never have isolated zeroes, in contrast to the one-variable case.

Exercise 2. Let $p(z, w)$ be a polynomial in two variables. Show that if the zero set of p is compact, then p is constant.

1.2 Integral representations

The one-variable Cauchy integral formula for a holomorphic function f inside a simple closed curve C says that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\, dw \qquad \text{for } z \text{ inside } C.$$

A remarkable feature of this formula is that the kernel $(w - z)^{-1}$ is both universal (independent of the domain) and holomorphic in the free variable. There is no such formula in higher dimensions! There are integral representations with a holomorphic kernel, but they depend on the domain, and there is a universal integral representation, but its kernel is not holomorphic. There is a huge literature about constructing and analyzing integral representations for various special types of domains.

For the special case of a polydisc, one can simply iterate the Cauchy integral. A reasonable working definition of "holomorphic function" is a function on a domain in $\mathbb{C}^n$ that is holomorphic in each variable separately and continuous in all variables jointly. If f is holomorphic in this sense on the closed unit polydisc, then iterating the Cauchy integral shows that

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|w_1|=1} \cdots \int_{|w_n|=1} \frac{f(w_1, \dots, w_n)}{(w_1 - z_1)\cdots(w_n - z_n)}\, dw_1 \dots dw_n$$

when the point z with coordinates $(z_1, \dots, z_n)$ is in the interior of the polydisc. (The assumed continuity of f guarantees that this integral makes sense and can be evaluated in any order by Fubini's theorem.) By the same arguments as in the single-variable case, this iterated Cauchy formula suffices to establish standard local properties of holomorphic functions. For example, holomorphic functions are infinitely differentiable, satisfy the Cauchy-Riemann equations in each variable, obey a local maximum principle, and admit local power series expansions. Moreover, a normal limit of holomorphic functions is holomorphic.

Exercise 3. Prove a multi-dimensional version of Hurwitz's theorem: the normal limit of nowhere-zero holomorphic functions is either nowhere zero or identically zero.

1.3 Partial differential equations

The one-dimensional Cauchy-Riemann equations are a pair of real partial differential equations for a pair of functions (the real and imaginary parts of a holomorphic function). In $\mathbb{C}^n$, there are still two functions, but there are $2n$ equations. Thus when $n > 1$, the inhomogeneous Cauchy-Riemann equations form an overdetermined system; hence there is a necessary compatibility condition for solvability. This feature is a significant difference from the one-variable theory.

When the inhomogeneous Cauchy-Riemann equations are solvable in $\mathbb{C}^2$ (or in higher dimension), there is a solution with compact support in case of compactly supported data. When $n = 1$, however, it is not always possible to solve the inhomogeneous Cauchy-Riemann equations while maintaining compact support. The Hartogs phenomenon can be interpreted as a manifestation of this difference.

1.4 Geometry

According to the one-variable Riemann mapping theorem, every bounded simply connected planar domain is biholomorphically equivalent to the unit disc. In higher dimension, there is no such simple topological classification of biholomorphically equivalent domains. Indeed, the unit ball in $\mathbb{C}^2$ and the unit bidisc in $\mathbb{C}^2$ are holomorphically inequivalent domains.

One way to understand intuitively why the situation changes in dimension 2 is to realize that there is extra room in the tangent space. In $\mathbb{C}^2$, there is room for one-dimensional complex analysis to happen in the tangent space to the boundary of a domain. Indeed, the boundary of the bidisc contains pieces of one-dimensional complex affine subspaces, while the boundary of the two-dimensional ball does not contain any such *analytic disc*.

Similarly, the zero set of a (not identically zero) holomorphic function in $\mathbb{C}^2$ is a one-dimensional complex variety, while the zero set of a holomorphic function in $\mathbb{C}^1$ is a zero-dimensional variety (that is, a discrete set of points).

There is a mismatch between the dimension of the domain and the dimension of the range of a multi-variable holomorphic function. One might, however, expect an equidimensional holomorphic *mapping* to be analogous to a one-variable holomorphic function. Here too there are surprises. For instance, there exists a biholomorphic mapping from all of $\mathbb{C}^2$ onto a proper subset of $\mathbb{C}^2$ whose complement has interior points. Such a mapping is called a *Fatou-Bieberbach map*.

Exercise 4. The image of a Fatou-Bieberbach map cannot have a bounded complement.

2 Power series

Examples in the introduction showed that the domain of convergence of a multi-variable power series can have various shapes; in particular, the domain need not be a convex set. Nonetheless, there is a special kind of convexity property that characterizes convergence domains.

Developing the theory requires some notation. A point $(z_1, \ldots, z_n)$ in $\mathbb{C}^n$ may be abbreviated simply by z. If α is a point of $\mathbb{C}^n$ whose coordinates are all non-negative integers, then z^α means the product $z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, the notation $\alpha!$ abbreviates the product $\alpha_1! \ldots \alpha_n!$, and $|\alpha|$ means $\alpha_1 + \cdots + \alpha_n$. A multi-variable power series can be written in the form $\sum_\alpha c_\alpha z^\alpha$ using this "multi-index" notation.

There is a little awkwardness in talking about convergence of a multi-variable power series $\sum_\alpha c_\alpha z^\alpha$, because convergence of a series depends (in general) on the order of summation, and there is no canonical ordering of n-tuples of non-negative integers when $n > 1$. For instance, the series $\sum_\alpha c_\alpha z^\alpha$ might mean either $\lim_{k\to\infty} \sum_{j=0}^k \sum_{|\alpha|=j} c_\alpha z^\alpha$ or $\lim_{k\to\infty} \sum_{\alpha_1=0}^k \cdots \sum_{\alpha_n=0}^k c_\alpha z^\alpha$. In general, these two limits need not be equal. It is convenient to restrict attention to *absolute* convergence, thereby eliminating any concern about the order of terms.

2.1 Domain of convergence

The *domain of convergence* of a power series is the interior of the set of points at which the series converges absolutely.[1] For example, the set where the two-variable power series $\sum_{n=1}^\infty z^n w^{n!}$ converges absolutely is the union of three sets: the points (z, w) for which $|w| < 1$ and z is arbitrary, the points $(0, w)$ for arbitrary w, and the points (z, w) for which $|w| = 1$ and $|z| < 1$. The domain of convergence is the first of these sets.

Since convergence domains are defined by considering absolute convergence, it is obvious that every convergence domain is *multi-circular*: if a point $(z_1, \ldots, z_n)$ is in the domain, then so is every point $(\lambda_1 z_1, \ldots, \lambda_n z_n)$ such that $|\lambda_1| = \cdots = |\lambda_n| = 1$. Moreover, a simple application of the comparison test for convergence of series shows that the point $(\lambda_1 z_1, \ldots, \lambda_n z_n)$ is still in the convergence domain when $|\lambda_j| \leq 1$ for each j. Thus *every convergence domain is a union of polydiscs centered at the origin.*

By expanding the Cauchy kernel in a power series, one finds from the iterated Cauchy formula (just as in the one-variable case) that a function holomorphic in a polydisc, or in a union of polydiscs with a common center, admits a power series expansion that converges in the (open) polydisc (or in the union of polydiscs).

[1]Usually the domain of convergence is assumed implicitly to be non-void. Thus one would not ordinarily speak of the domain of convergence of the series $\sum_{n=1}^\infty n!\, z^n w^n$.

A multi-circular domain is also called a Reinhardt domain. A Reinhardt domain is called *complete* if whenever a point z is in the domain, then the polydisc $\{ w : |w_1| \le |z_1|, \dots, |w_n| \le |z_n| \}$ is in the domain too. Thus the preceding discussion can be rephrased as saying that every convergence domain is a complete Reinhardt domain, and every holomorphic function defined in a complete Reinhardt domain can be represented there by a convergent power series.

More is true, however. If both $\sum_\alpha |c_\alpha z^\alpha|$ and $\sum_\alpha |c_\alpha w^\alpha|$ converge, then Hölder's inequality implies that $\sum_\alpha |c_\alpha||z^\alpha|^t|w^\alpha|^{1-t}$ converges when $0 \le t \le 1$. This means that a convergence domain is *logarithmically convex*. Since a convergence domain is complete and multi-circular, the domain is determined by the points with positive real coordinates; replacing the coordinates of each such point by their logarithms produces a convex domain in $\mathbb{R}^n$.

2.2 Characterization of domains of convergence

The following theorem[2] gives a geometric characterization of domains of convergence of power series.

Theorem 1. *A complete Reinhardt domain in $\mathbb{C}^n$ is the domain of convergence of some power series if and only if the domain is logarithmically convex.*

Proof. The preceding discussion shows that a convergence domain is necessarily logarithmically convex. What remains to prove is that if D is a logarithmically convex complete Reinhardt domain, then there exists some power series $\sum_\alpha c_\alpha z^\alpha$ whose domain of convergence is D. Let us assume at first that the domain D is bounded, for the idea of the construction is easier to see in that case.

Let $\|z^\alpha\|_D$ denote $\sup\{ |z^\alpha| : z \in D \}$; this expression is finite when D is bounded. The claim is that $\sum_\alpha z^\alpha/\|z^\alpha\|_D$ is the desired power series whose domain of convergence is equal to D. What needs to be checked is that this series converges absolutely at each point of D, and the series fails to converge absolutely in a neighborhood of any point outside the closure of D.

If w is a particular point in the interior of D, then w is in the interior of some open polydisc of polyradius r contained in D. Let λ denote $\max_{1\le j\le n} |w_j|/r_j$. Then $\lambda < 1$, and $|w^\alpha|/\|z^\alpha\|_D \le \lambda^{|\alpha|}$. Therefore the series $\sum_\alpha w^\alpha/\|z^\alpha\|_D$ converges absolutely by comparison with the convergent dominating series $\sum_\alpha \lambda^{|\alpha|}$. Thus the first half of the claim is valid.

To check the second half of the claim, suppose (seeking a contradiction) that the convergence domain of the series $\sum_\alpha z^\alpha/\|z^\alpha\|_D$ contains a certain point w that lies outside the closure of D. Since convergence domains are (by definition) open sets, there is no loss of generality in assuming that every coordinate of the point w is nonzero.

[2] Fritz Hartogs, Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten, *Mathematische Annalen* **62** (1906), no. 1, 1–88. (Hartogs considered domains in $\mathbb{C}^2$.)

Moreover, the convergence domain is multi-circular, so there is no loss of generality in assuming that every coordinate of w is a positive real number.

The hypothesis that D is logarithmically convex means that the set

$$\log D := \{\, (u_1, \dots, u_n) \in \mathbb{R}^n : (e^{u_1}, \dots, e^{u_n}) \in D \,\}$$

is a convex set in $\mathbb{R}^n$. The point $(\log w_1, \dots, \log w_n)$ can be separated from this convex set by a hyperplane. Equivalently, there is a linear function $h : \mathbb{R}^n \to \mathbb{R}$ whose value at the point $(\log w_1, \dots, \log w_n)$ exceeds the supremum of h over the convex set $\log D$. Suppose that $h(u) = \beta_1 u_1 + \cdots + \beta_n u_n$, where the β_j are certain real numbers.

Since D contains a neighborhood of the origin, there is a real constant m (possibly negative) such that the convex set $\log D$ contains all points u for which $\max_{1\le j\le n} u_j \le m$. Therefore none of the numbers β_j can be negative, for otherwise the function h would take arbitrarily large positive values on $\log D$. Moreover, since D is bounded, there is a real constant M such that $\log D$ is contained in the set of all points u such that $\max_{1\le j\le n} u_j \le M$. Consequently, if each number β_j is increased by some small positive amount ϵ_j, then the supremum of h over $\log D$ increases by no more than $nM \max_{1\le j\le n} \epsilon_j$. Thus, the coefficients of h can be perturbed slightly, and h will remain a separating function. Accordingly, there is no loss of generality in assuming that each β_j is a positive rational number. Multiplying by a common denominator shows that the coefficients β_j can be taken to be positive integers.

Exponentiating to get back to the space $\mathbb{C}^n$ produces a certain multi-index β such that $|w^\beta| > \|z^\beta\|_D$. It follows that $|w^{k\beta}| > \|z^{k\beta}\|_D$ for every positive integer k. Consequently, the series $\sum_\alpha w^\alpha/\|z^\alpha\|_D$ diverges, for there are infinitely many terms of modulus larger than 1. This conclusion contradicts the supposition that the series converges in a neighborhood of a point w outside the closure of D, so D must be the convergence domain of the series after all.

To complete the proof, one has to handle the case of unbounded domains. When D is unbounded, let D_r denote the intersection of D with the ball of radius r centered at the origin. Then D_r is a bounded, complete, logarithmically convex Reinhardt domain, so the preceding analysis applies to D_r. It will not work, however, to splice together power series of the type just constructed for an increasing sequence of values of r, for none of these series will converge at every point of the unbounded domain D.

One way to finish the argument (and to advertise coming attractions) is to apply a famous theorem of H. Behnke and K. Stein (usually called *the* Behnke-Stein theorem), according to which an increasing union of domains of holomorphy is again a domain of holomorphy.[3] The next section will show that a convergence domain for a power series supports some (other) power series that cannot be analytically continued across any boundary point whatsoever. Hence each D_r is a domain of holomorphy, and the Behnke-Stein theorem implies that D is a domain of holomorphy. Thus D supports some holomorphic function that cannot be analytically continued across any boundary point of D. Since D is a complete Reinhardt domain, this holomorphic function is represented

[3] H. Behnke and K. Stein, Konvergente Folgen von Regularitätsbereichen und die Meromorphiekonvexität, *Mathematische Annalen* **116** (1938) 204–216.

by a power series that converges in all of D, and evidently D is the convergence domain of this power series.

The argument in the preceding paragraph is unsatisfying because, besides being anachronistic and not self-contained, it provides no concrete construction of the required power series. Here is an alternate argument that is nearly concrete.

Consider the countable set of points outside the closure of D that have positive rational coordinates. Make a list $\{w(j)\}_{j=1}^{\infty}$ of these points in which each point appears infinitely often. Since the domain D_j is bounded, the first part of the proof provides a multi-index $\beta(j)$ of positive integers such that $w(j)^{\beta(j)} > \|z^{\beta(j)}\|_{D_j}$. Multiplying this multi-index by a positive integer gives another multi-index with the same property, so it may be assumed that $|\beta(j+1)| > |\beta(j)|$ for every j. The claim is that

$$\sum_{j=1}^{\infty} \frac{z^{\beta(j)}}{\|z^{\beta(j)}\|_{D_j}}$$

is a power series whose domain of convergence is D.

First of all, the indicated series *is* a power series, since no two of the multi-indices $\beta(j)$ are equal (so there are no common terms to combine). If z is an interior point of D, then z is inside the bounded domain D_k for sufficiently large k. Therefore the tail of the series is dominated by $\sum_\alpha |z^\alpha|/\|z^\alpha\|_{D_k}$, and the latter series converges inside D_k by the argument in the first part of the proof. Thus the convergence domain of the indicated series is at least as large as D.

On the other hand, if the series were to converge absolutely in some neighborhood of a point outside D, then the series would converge at some point ζ outside the closure of D having positive rational coordinates. Since there are infinitely many values of j for which $w(j) = \zeta$, the series

$$\sum_{j=1}^{\infty} \frac{\zeta^{\beta(j)}}{\|z^{\beta(j)}\|_{D_j}}$$

has (by construction) infinitely many terms larger than 1, and so diverges. Thus the convergence domain of the constructed series is no larger than D.

Thus every logarithmically convex, complete, multi-circular domain, whether bounded or unbounded, is the domain of convergence of some power series. □

2.3 Natural boundaries

Although the one-dimensional power series $\sum_{k=0}^{\infty} z^k$ has the unit disc as its convergence domain, the function represented by the series, which is $1/(1-z)$, extends holomorphically across most of the boundary. On the other hand, there exist power series that converge in the unit disc and have the unit circle as "natural boundary." (One example is the gap series $\sum_k z^{2^k}$.) The following theorem says that also in higher dimensions, every convergence domain (that is, every logarithmically convex, complete Reinhardt domain) is the natural domain of existence of some holomorphic function.

Theorem 2. *The domain of convergence of a power series is a domain of holomorphy. More precisely, for every domain of convergence there exists some power series that converges in the domain and that is singular at every boundary point.*

Recall that the word "singular" does not necessarily mean that the function blows up. To say that a power series is singular at a boundary point of its domain of convergence means that the series does not admit a direct analytic continuation to a neighborhood of the point. A function whose modulus tends to infinity is singular, but so is a function whose modulus tends to zero exponentially fast.

Proof of Theorem 2. Let D be the domain of convergence (assumed nonvoid) of the power series $\sum_\alpha c_\alpha z^\alpha$. Since the two series $\sum_\alpha c_\alpha z^\alpha$ and $\sum_\alpha |c_\alpha| z^\alpha$ have the same region of absolute convergence, there is no loss of generality in assuming from the outset that the coefficients c_α are non-negative real numbers.

The topology of uniform convergence on compact sets is metrizable, so the space of holomorphic functions on D is a complete metric space. Hence the Baire category theorem is available. The goal is to prove that the holomorphic functions on D that extend holomorphically across some boundary point form a set of first category in this metric space. That will imply the existence of power series that are singular at every boundary point of D; indeed, most power series that converge in D will have this property.

A first step toward the goal is a multi-dimensional version of an observation that dates back to the end of the nineteenth century.

Lemma 1 *(Multi-dimensional Pringsheim lemma).* A power series $\sum_\alpha c_\alpha z^\alpha$ with real non-negative coefficients c_α is singular at every boundary point $(r_1, \dots, r_n)$ of the domain of convergence at which all the coordinates r_j are positive real numbers.

Proof. Seeking a contradiction, suppose that the holomorphic function f represented by the series extends holomorphically to a neighborhood of some boundary point r having positive coordinates. Consider the Taylor series of f about the interior point $\frac{1}{2}r$:

$$f(z) = \sum_\alpha \frac{1}{\alpha!} f^{(\alpha)}(\tfrac{1}{2}r)(z - \tfrac{1}{2}r)^\alpha.$$

By the assumption, this series converges when $z = r + \epsilon\mathbf{1}$, where $\mathbf{1} = (1, \dots, 1)$, and ϵ is a sufficiently small positive number. Differentiating the original series shows that

$$f^{(\alpha)}(\tfrac{1}{2}r) = \sum_{\beta \geq \alpha} \frac{\beta!}{(\beta - \alpha)!} c_\beta (\tfrac{1}{2}r)^{\beta - \alpha}.$$

Combining these two expressions shows that the series

$$\sum_\alpha \sum_{\beta \geq \alpha} \binom{\beta}{\alpha} c_\beta (\tfrac{1}{2}r)^{\beta - \alpha} (\tfrac{1}{2}r + \epsilon\mathbf{1})^\alpha$$

converges. Since all the quantities involved are non-negative real numbers, the order of summation can be interchanged without affecting the convergence; the expression simplifies to the series

$$\sum_{\beta} c_\beta (r + \epsilon \mathbf{1})^\beta.$$

This series is the original series for f, now seen to be absolutely convergent in a neighborhood of the point r. Hence r could not have been a boundary point of the domain of convergence. The contradiction shows that f must have been singular at r after all. ☐

In view of the lemma, the power series $\sum_\alpha c_\alpha z^\alpha$ (now assumed to have non-negative coefficients) is singular at all the boundary points of the domain of convergence having positive real coordinates. If $(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})$ is an arbitrary boundary point having all coordinates non-zero, then the power series $\sum_\alpha c_\alpha e^{-i(\alpha_1\theta_1 + \cdots + \alpha_n\theta_n)} z^\alpha$ is singular at this boundary point. In other words, for every boundary point having non-zero coordinates, there exists some power series that converges in D but is singular at that boundary point.

Now choose a countable dense subset $\{p_j\}$ of the boundary of D consisting of points with non-zero coordinates. The space of holomorphic functions on $D \cup B(p_j, 1/k)$ embeds continuously into the space of holomorphic functions on D. The image of the embedding is not the whole space, for the preceding discussion produced a power series that does not extend into $B(p_j, 1/k)$. Consequently, by the first theorem of Chapter 3 of Stefan Banach's famous book *Théorie des opérations linéaires*, the image of the embedding must be of first category (the cited theorem says that if the image were of second category, then it would be the whole space, which it is not). Thus the set of power series on D that extend some distance across some boundary point is a countable union of sets of first category, hence itself a set of first category. Thus most power series that converge in D have the boundary of D as natural boundary. ☐

2.4 Summary

The preceding discussion shows that for complete Reinhardt domains, the following properties are all equivalent.

- The domain is logarithmically convex.
- The domain is the domain of convergence of some power series.
- The domain is a domain of holomorphy.

In other words, the problem of characterizing domains of holomorphy is solved for the special case of complete Reinhardt domains.

3 Convexity

From one point of view, convexity is an unnatural property in complex analysis. The Riemann mapping theorem shows that already in dimension 1, convexity is not preserved by biholomorphic mappings. The unit disc is conformally equivalent to any other bounded simply connected domain, convex or not.

On the other hand, section 2.2 revealed that a special kind of convexity—logarithmic convexity—appears naturally in studying convergence domains of power series. Studying various analogues of convexity has been a fruitful approach to solving some fundamental problems in multi-dimensional complex analysis.

3.1 Real convexity

Ordinary geometric convexity can be described either through an internal geometric property—the line segment joining two points of the set stays within the set—or through an external property—every point outside the set can be separated from the set by a hyperplane. The latter property can be rephrased in analytic terms by saying that every point outside the set can be separated from the set by a linear function; that is, there is a linear function that is larger at the specified exterior point than anywhere on the set.

For an arbitrary set, not necessarily convex, its *convex hull* is the smallest convex set containing it, that is, the intersection of all convex sets containing it.

Observe that an open set G is convex if and only if the convex hull of every compact subset K is again a compact subset of G. Indeed, if K is a subset of G, then the convex hull of K is a subset of the convex hull of G, so if G is already convex, then the convex hull of K is a subset of G; moreover, the convex hull of a compact set evidently is compact. Conversely, if G is not convex, then there are two points of G such that the line segment joining them goes outside of G; take K to be the union of the two points.

3.2 Convexity with respect to a class of functions

The analytic description of convexity has a natural generalization. Suppose that $\mathcal{F}$ is a class of upper semi-continuous real-valued functions on an open set G in $\mathbb{C}^n$ (which might be $\mathbb{C}^n$ itself). [Recall that a real-valued function f is upper semi-continuous if $f^{-1}(-\infty, a)$ is an open set for every real number a; an upper semi-continuous function attains a maximum on a compact set.] A compact subset K of G is called convex with respect to $\mathcal{F}$ if for every point p of $G \setminus K$ there exists an element f of $\mathcal{F}$ for which $f(p) > \max_{z \in K} f(z)$; in other words, every point outside K can be separated from K by a function in $\mathcal{F}$. If $\mathcal{F}$ is a class of functions that are complex-valued but not real-valued (typically holomorphic functions), then it is natural to consider convexity with respect to the class of absolute values of the functions in $\mathcal{F}$; one typically says simply "$\mathcal{F}$-convex" for short when the meaning is really "$\{\,|f| : f \in \mathcal{F}\,\}$-convex".

The $\mathcal{F}$-convex hull of a compact set K, denoted by $\widehat{K}_{\mathcal{F}}$ (or simply $\widehat{K}$ if $\mathcal{F}$ is understood) is the smallest $\mathcal{F}$-convex set containing K. One says that an open set Ω is $\mathcal{F}$-convex if for every compact subset K, the $\mathcal{F}$-convex hull $\widehat{K}_{\mathcal{F}}$ is again a compact subset of Ω.

Example 1. Let G be $\mathbb{R}^n$, and let $\mathcal{F}$ be the set of all continuous functions. Then every compact set K is $\mathcal{F}$-convex because, by Urysohn's lemma, every point not in K can be separated from K by a continuous function.

Example 2. Let G be $\mathbb{C}^n$, and let $\mathcal{F}$ be the set of coordinate functions, $\{z_1, \dots, z_n\}$. The $\mathcal{F}$-convex hull of a single point w is the set of all points z for which $|z_j| \le |w_j|$ for all j, that is, the polydisc determined by the point w. (If some coordinate of w is equal to 0, then the polydisc is degenerate.) More generally, the $\mathcal{F}$-convex hull of a compact set K is the set of points z for which $|z_j| \le \max\{\,|\zeta_j| : \zeta \in K\,\}$ for every j. Consequently, the $\mathcal{F}$-convex open sets are precisely the open polydiscs.

A useful observation is that increasing the class of functions $\mathcal{F}$ makes it easier to separate points, so the collection of $\mathcal{F}$-convex sets becomes larger. In other words, if $\mathcal{F}_1 \subset \mathcal{F}_2$, then every $\mathcal{F}_1$-convex set is also $\mathcal{F}_2$-convex.

Exercise 5. In $\mathbb{R}^n$, convexity with respect to the class of linear functions $a_1x_1+\cdots+a_nx_n$ is the same as ordinary geometric convexity. It is the same thing to consider convexity with respect to the class of affine linear functions $a_0 + a_1x_1 + \cdots + a_nx_n$.

1. Suppose $\mathcal{F}$ is the set $\{\,|a_1x_1 + \cdots + a_nx_n|\,\}$ of absolute values of linear functions. Describe the $\mathcal{F}$-convex hull of a compact set.

2. Suppose $\mathcal{F}$ is the set $\{\,|a_0 + a_1x_1 + \cdots + a_nx_n|\,\}$ of absolute values of affine linear functions. Describe the $\mathcal{F}$-convex hull of a compact set.

Exercise 6. Repeat the preceding exercise in the setting of $\mathbb{C}^n$ and functions with complex coefficients:

1. Suppose $\mathcal{F}$ is the set $\{\,|a_1z_1 + \cdots + a_nz_n|\,\}$ of absolute values of complex linear functions. Describe the $\mathcal{F}$-convex hull of a compact set.

2. Suppose $\mathcal{F}$ is the set $\{\,|a_0+a_1z_1+\cdots+a_nz_n|\,\}$ of absolute values of affine complex linear functions. Describe the $\mathcal{F}$-convex hull of a compact set.

Observe that a point and a compact set can be separated by $|f|$ if and only they can be separated by $|f|^2$ or more generally by $|f|^k$. Hence there is no loss of generality in assuming that a class $\mathcal{F}$ of holomorphic functions is closed under forming positive integral powers. Allowing arbitrary products, however, changes the situation, as the next example demonstrates.

Example 3. Let G be all of $\mathbb{C}^n$, and let $\mathcal{F}$ be the set of monomials z^α. As in the preceding example, the $\mathcal{F}$-convex hull of a point is the polydisc determined by the point. Hence the $\mathcal{F}$-convex hull of an open set must be a complete Reinhardt domain. Moreover, the $\mathcal{F}$ convex hull of a two-point set $\{w, \zeta\}$ evidently contains all points z for which there exists a number t between 0 and 1 such that $|z_j| \le |w_j|^t |\zeta_j|^{1-t}$ for all j. Therefore an $\mathcal{F}$-convex open set must be a logarithmically, convex complete Reinhardt domain.

Exercise 7. Show that conversely, a logarithmically convex, complete Reinhardt domain is convex with respect to the class $\mathcal{F}$ consisting of the monomials z^α.
Hint: see the proof of Theorem 1.

3.2.1 Polynomial convexity

Again let G be all of $\mathbb{C}^n$, and let $\mathcal{F}$ be the set of (holomorphic) polynomials. Then $\mathcal{F}$-convexity is called *polynomial convexity*. (When the setting is $\mathbb{C}^n$, the word "polynomial" usually means "holomorphic polynomial", that is, a polynomial in the complex coordinates $z_1, \ldots, z_n$ rather than a polynomial in the underlying real coordinates.)

A first observation is that the polynomial hull of a compact set is a subset of the ordinary convex hull. This fact follows from the solution to Exercise 6. Alternatively, one can argue that if a point is separated from a compact set by a real linear function $\operatorname{Re} \ell(z)$, then it is separated by $e^{\operatorname{Re} \ell(z)}$, and hence by $|e^{\ell(z)}|$; now the entire function $e^{\ell(z)}$ can be approximated uniformly on compact sets by polynomials.

When $n = 1$, polynomial convexity is characterized by a topological property. Recall Runge's approximation theorem, which says that if K is a compact subset of $\mathbb{C}$ (not necessarily connected), and if K has no holes (that is, $\mathbb{C} \setminus K$ is connected), then every function that is holomorphic in a neighborhood of K can be approximated uniformly on K by (holomorphic) polynomials.[1] Now if K has no holes, and p is a point outside K, then Runge's theorem implies that the function equal to 0 in neighborhood of K and equal to 1 in a neighborhood of p can be arbitrarily well approximated on $K \cup \{p\}$ by polynomials; hence p is not in the polynomial hull of K. On the other hand, if K has a hole, then the maximum principle implies that points inside the hole are in the polynomial hull of K. In other words, a compact set K in $\mathbb{C}$ is polynomially convex if and only if K has no holes, that is, $\mathbb{C} \setminus K$ is connected. A connected open subset of $\mathbb{C}$ is polynomially convex if and only if it is simply connected, that is, its complement with respect to the extended complex numbers is connected.

The story is much more complicated when $n > 1$, for then polynomial convexity is no longer determined by a topological condition. For instance, whether or not a circle (of real dimension 1) is polynomially convex depends on how the curve is situated with respect to the complex structure of $\mathbb{C}^n$.

Example 4. (a) In $\mathbb{C}^2$, the circle $\{\, (\cos\theta + i \sin\theta, 0) : 0 \le \theta \le 2\pi \,\}$ is not polynomially convex. Indeed, the (one-dimensional) maximum principle implies that the polynomial hull of this curve is the disc $\{\, (z_1, 0) : |z_1| \le 1 \,\}$.

[1] There is a deeper theorem due to S. N. Mergelyan: it suffices if the function to be approximated is holomorphic at the interior points of K and continuous on K.

(b) In $\mathbb{C}^2$, the circle $\{ (\cos\theta, \sin\theta) : 0 \le \theta \le 2\pi \}$ is polynomially convex. Indeed, since the polynomial hull is a subset of the ordinary convex hull, one need only show that points inside the disc bounded by the circle can be separated from the circle by (holomorphic) polynomials. The polynomial $1 - z_1^2 - z_2^2$ is identically equal to 0 on the circle and takes positive real values at points inside the circle, so this polynomial exhibits the required separation.

The preceding idea can easily be generalized to produce a wider class of examples of polynomially convex sets.

Example 5. If K is a compact subset of the real subspace of $\mathbb{C}^n$ (that is, $K \subset \mathbb{R}^n \subset \mathbb{C}^n$), then K is polynomially convex.

To see why, first notice that convexity with respect to (holomorphic) polynomials is the same property as convexity with respect to entire functions, since an entire function can be approximated uniformly on a compact set by polynomials (for instance, by the partial sums of the Maclaurin series). Therefore it suffices to write down an entire function whose modulus separates K from a specified point q outside of K.

A function that does the trick is $\exp \sum_{j=1}^n -(z_j - \operatorname{Re} q_j)^2$. For let $M(z)$ denote the modulus of this function: namely, $\exp \sum_{j=1}^n [(\operatorname{Im} z_j)^2 - (\operatorname{Re} z_j - \operatorname{Re} q_j)^2]$. If $q \notin \mathbb{R}^n$, then $M(q) = \exp \sum_{j=1}^n (\operatorname{Im} q_j)^2 > 1$, and $\max_{z\in K} M(z) = \max_{z\in K} \exp \sum_{j=1}^n -(z_j - \operatorname{Re} q_j)^2 \le 1$. On the other hand, if $q \in \mathbb{R}^n$ but $q \notin K$, then the expression $\sum_{j=1}^n (z_j - \operatorname{Re} q_j)^2$ has a positive lower bound on the compact set K, so $\max_{z\in K} M(z) < 1$, while $M(q) = 1$. The required separation holds in both cases. (Actually, it is enough to check the second case, for the polynomial hull of K is a subset of the convex hull of K and hence a subset of $\mathbb{R}^n$.)

Exercise 8. Show that every compact subset of a *totally real* subspace of $\mathbb{C}^n$ is polynomially convex. (A subspace is called totally real if it contains no complex line. In other words, a subspace is totally real if, whenever z is a nonzero point in the subspace, the point iz is not in the subspace.)

Having some polynomially convex sets in hand, one can generate more by the following example.

Example 6. If K is a polynomially convex subset of $\mathbb{C}^n$, and p is a polynomial, then the graph $\{ (z, p(z)) \in \mathbb{C}^{n+1} : z \in K \}$ is a polynomially convex subset of $\mathbb{C}^{n+1}$.

For suppose $\alpha \in \mathbb{C}^n$ and $\beta \in \mathbb{C}$, and (α, β) is not in the graph of p over K; to separate the point (α, β) from the graph by a polynomial, consider two cases. If $\alpha \notin K$, then there is a polynomial of n variables that separates α from K in $\mathbb{C}^n$; the same polynomial, viewed as a polynomial on $\mathbb{C}^{n+1}$ that is independent of z_{n+1}, separates the point (α, β) from the graph of p. Suppose, on the other hand, that $\alpha \in K$ and $\beta \ne p(\alpha)$. Then the polynomial $z_{n+1} - p(z)$ is identically equal to 0 on the graph and is not equal to 0 at (α, β), so this polynomial separates (α, β) from the graph.

Exercise 9. If f is a function that is continuous on the closed unit disc in $\mathbb{C}$ and holomorphic on the interior of the disc, then the graph of f in $\mathbb{C}^2$ is polynomially convex.

More generally, a smooth analytic disc (the image in $\mathbb{C}^n$ of a holomorphic embedding of the closed unit disc whose derivative is never zero) is always polynomially convex.[2] Biholomorphic images of polydiscs, however, can fail to be polynomially convex.[3]

The basic examples of polynomially convex sets with interior are the polynomial polyhedra. A compact set K in $\mathbb{C}^n$ is called a *polynomial polyhedron* if there are finitely many polynomials such that $K = \{ z \in \mathbb{C}^n : |p_1(z)| \le 1, \ldots, |p_k(z)| \le 1 \}$ (the model case being the polydisc $\{ z \in \mathbb{C}^n : |z_1| \le 1, \ldots, |z_n| \le 1 \}$). Such a set evidently is polynomially convex, since a point in the complement is separated from K by at least one of the defining polynomials. Notice that the number k of polynomials can be much larger than the dimension n. (If K is compact and non-empty, then the number k cannot be less than n, but proving this property requires some additional tools.[4])

A bounded open set G is called a polynomial polyhedron if there are finitely many polynomials such that $G = \{ z \in \mathbb{C}^n : |p_1(z)| < 1, \ldots, |p_k(z)| < 1 \}$. A standard way to force G to be bounded is to include in the set of defining polynomials the functions z_j/R for some large R and for each j from 1 to n. An open polynomial polyhedron G is evidently polynomially convex, since on a compact subset of G, the functions $|p_j|$ are all bounded above by some number less than 1.

The following theorem says that any polynomially convex set can be approximated by polynomial polyhedra.

Theorem 3. *(a) If K is a compact polynomially convex set, and U is an open neighborhood of K, then there is an open polynomial polyhedron P such that $K \subset P \subset U$.*

(b) If G is a polynomially convex open set, then G can be written as the union of an increasing sequence of open polynomial polyhedra.

Proof. (a) Since the set K is bounded, it is contained in the interior of some closed polydisc D. For each point w in $D \setminus U$, there is a polynomial p that separates w from K. Multiplying p by a suitable constant, one can arrange that $\max\{ |p(z)| : z \in K \} < 1 < |p(w)|$. Hence the set $\{ z : |p(z)| < 1 \}$ contains K and is disjoint from a neighborhood of w. Since the set $D \setminus U$ is compact, there are finitely many polynomials $p_1, \ldots, p_k$ such that the polyhedron $\bigcap_{j=1}^{k} \{ z : |p_j(z)| < 1 \}$ contains K and does not intersect $D \setminus U$. Intersecting this polyhedron with D gives a new polyhedron, and this new polyhedron contains K and is contained in U.

(b) Exhaust G by an increasing sequence of compact sets. The polynomial hulls of these sets form another increasing sequence of compact subsets of G (since G is

[2] John Wermer, An example concerning polynomial convexity, *Mathematische Annalen* **139** (1959) 147–150.

[3] For an example in $\mathbb{C}^3$, see John Wermer, Addendum to "An example concerning polynomial convexity", *Mathematische Annalen* **140** (1960) 322–323. For an example in $\mathbb{C}^2$, see John Wermer, On a domain equivalent to the bidisc, *Mathematische Annalen* **248** (1980), no. 3, 193–194.

[4] If w is a point of K, then the k sets $\{z \in \mathbb{C}^n : p_j(z) - p_j(w) = 0\}$ are analytic varieties of codimension 1 that intersect in an analytic variety of dimension at least $n-k$ that is contained in K. If $k < n$, then this analytic variety has positive dimension, but there are no compact analytic varieties of positive dimension.

polynomially convex). After possibly omitting some of the sets and renumbering, one obtains a sequence $\{K_j\}_{j=1}^{\infty}$ of polynomially convex compact subsets of G such that each K_j is contained in the interior of K_{j+1}. The first part of the theorem then provides a sequence of open polynomial polyhedra P_j such that $K_j \subset P_j \subset K_{j+1}$. $\square$

Although the theory of polynomial convexity is sufficiently mature that there exists a good reference book,[5] it remains fiendishly difficult to determine the polynomial hull of even quite simple sets. Here is one tractable example: *If K_1 and K_2 are disjoint, compact, convex sets in $\mathbb{C}^n$, then the union $K_1 \cup K_2$ is polynomially convex.*

Proof. The convex sets K_1 and K_2 can be separated by a real hyperplane, or equivalently by the real part of a complex linear function ℓ. The geometric picture is that ℓ projects $\mathbb{C}^n$ onto a complex line (a one-dimensional complex subspace). Then $\ell(K_1)$ and $\ell(K_2)$ can be viewed as disjoint compact convex sets in $\mathbb{C}$.

Suppose now that w is a point outside of $K_1 \cup K_2$. If $\ell(w) \notin \ell(K_1) \cup \ell(K_2)$, then Runge's theorem provides a polynomial p of one variable such that $|p(\ell(w))| > 1$, and $|p(z)| < 1$ when $z \in \ell(K_1) \cup \ell(K_2)$. In other words, the polynomial $p \circ \ell$ separates w from $K_1 \cup K_2$ in $\mathbb{C}^n$.

If, on the other hand, $\ell(w) \in \ell(K_1) \cup \ell(K_2)$, then one may as well assume that $\ell(w) \in \ell(K_1)$. Since $w \notin K_1$, however, and K_1 is polynomially convex, there is a polynomial p on $\mathbb{C}^n$ such that $|p(w)| > 1$ and $|p(z)| < 1/3$ when $z \in K_1$. Let M be an upper bound for $|p|$ on K_2. Applying Runge's theorem in $\mathbb{C}$ gives a polynomial q of one variable such that $|q| < 1/(3M)$ on $\ell(K_2)$ and $2/3 \leq |q| \leq 1$ on $\ell(K_1)$. The product polynomial $p \times (q \circ \ell)$ separates w from $K_1 \cup K_2$: for on K_1, the first factor has modulus less than $1/3$, and the second factor has modulus no greater than 1; on K_2, the first factor has modulus at most M, and the second factor has modulus less than $1/(3M)$; and at w, the modulus of the first factor exceeds 1, and the modulus of the second factor is at least $2/3$. $\square$

The preceding proposition is a special case of a separation lemma of Eva Kallin, who showed that the union of three closed, pairwise disjoint balls in $\mathbb{C}^n$ is always polynomially convex.[6] The question of the polynomial convexity of the union of four pairwise disjoint closed balls is still open after more than four decades. The problem is subtle, for Kallin also gave an example of three closed pairwise disjoint polydiscs in $\mathbb{C}^3$ whose union is *not* polynomially convex.

Runge's theorem in dimension 1 indicates that polynomial convexity is intimately connected with the approximation of holomorphic functions by polynomials. There is an analogue of Runge's theorem in higher dimension, known as the Oka-Weil theorem, to be revisited later. Here is the statement.

[5] Edgar Lee Stout, *Polynomial Convexity*, Birkhäuser Boston, 2007.

[6] Eva Kallin, Polynomial convexity: The three spheres problem, in *Proceedings of the Conference on Complex Analysis (Minneapolis, 1964)*, pp. 301–304, Springer, 1965.

Theorem 4 (Oka-Weil). *If K is a compact, polynomially convex set in $\mathbb{C}^n$, then every function holomorphic in a neighborhood of K can be approximated uniformly on K by (holomorphic) polynomials.*

Exercise 10. Give an example of a compact set K in $\mathbb{C}^2$ such that every function holomorphic in a neighborhood of K can be approximated uniformly on K by polynomials, yet K is not polynomially convex.

3.2.2 Linear and rational convexity

The preceding examples involved functions that are globally defined on the whole space independently of the region in question. In many interesting cases, however, the class of functions varies with the region.

Suppose that G is an open set in $\mathbb{C}^n$, and let $\mathcal{F}$ be the class of those linear fractional functions

$$\frac{a_0 + a_1 z_1 + \cdots + a_n z_n}{b_0 + b_1 z_1 + \cdots + b_n z_n}$$

that happen to be holomorphic on G (in other words, the denominator has no zeroes in G). The claim is that G is $\mathcal{F}$-convex if and only if through each boundary point of G there passes a complex hyperplane that does not intersect G. (A simple example of such a set G is $\mathbb{C}^2 \setminus \{\, (z_1, z_2) \in \mathbb{C}^n : z_2 = 0 \,\}$, where the boundary points of G form a complex line.)

Suppose first that G is $\mathcal{F}$-convex, and let w be a point in the boundary of G. If K is a compact subset of G, then $\widehat{K}_{\mathcal{F}}$ is again a compact subset of G, so to every point w' in G sufficiently close to w there corresponds a linear fractional function f in $\mathcal{F}$ such that $f(w') = 1 > \max\{\, |f(z)| : z \in K \,\}$. Let ℓ denote the difference between the numerator of f and the denominator of f; then $\ell(z) = 0$ at a point z in G if and only if $f(z) = 1$. Hence the zero set of ℓ, which is a complex hyperplane, passes through w' and does not intersect K. Multiply ℓ by a suitable constant to ensure that the vector consisting of the coefficients of ℓ has length 1.

Now exhaust G by an increasing sequence of compact sets K_j. The preceding construction produces a sequence of points w_j in G converging to w and a sequence of normalized first-degree polynomials ℓ_j such that $\ell_j(w_j) = 0$, and the zero set of ℓ_j does not intersect K_j. The set of vectors of length 1 is compact, so it is possible to pass to the limit of a suitable subsequence to obtain a complex hyperplane that passes through the boundary point w and does not intersect the open set G.

Conversely, a supporting complex hyperplane at a boundary point w is the zero set of a certain first-degree polynomial ℓ, and $1/\ell$ is then a linear fractional function that is holomorphic on G and blows up at w. Therefore the $\mathcal{F}$-convex hull of a compact set K in G stays away from w. Since w is arbitrary, the hull $\widehat{K}_{\mathcal{F}}$ is a compact subset of G. Since K is arbitrary, the domain G is $\mathcal{F}$-convex by definition.

A domain that is convex with respect to the linear fractional functions that are holomorphic on it is called *weakly linearly convex*. (A domain is called linearly convex if the complement can be written as a union of complex hyperplanes. The terminology is not completely standardized, however, so one has to check each author's definitions.)

Next consider general rational functions (quotients of polynomials). A compact set K is called *rationally convex* if every point w outside K can be separated from K by a rational function that is holomorphic on $K \cup \{w\}$, that is, if there is a rational function f such that $|f(w)| > \max\{\, |f(z)| : z \in K \,\}$. In this definition, it does not much matter whether or not f is holomorphic at w, for if $f(w)$ is undefined, then one can slightly perturb the coefficients of f to make $|f(w)|$ be a large finite number without changing the values of f on K very much.

Example 7. Every compact set K in $\mathbb{C}$ is rationally convex. Indeed, if w is a point outside K, then the rational function $1/(z-w)$ blows up at w, so w is not in the rationally convex hull of K. [For a suitably small positive ϵ, the rational function $1/(z-w-\epsilon)$ has larger modulus at w than it does anywhere on K.]

Exercise 11. The domain $\{\, (z_1, z_2) \in \mathbb{C}^2 : 1 < |z_1| < 2 \text{ and } 1 < |z_2| < 2 \,\}$ is rationally convex.

There is a certain awkwardness in talking about multi-variable rational functions, because the singularities can be either poles (where the modulus blows up) or points of indeterminacy (like the origin for the function z_1/z_2). Therefore it is convenient to rephrase the notion of rational convexity in a way that uses only polynomials.

The notion of polynomial convexity involves separation by the modulus of a polynomial; it is natural to introduce the modulus in order to write inequalities. One could, however, consider the weaker separation property that a point w is separated from a compact set K if there is a polynomial p such that the image of w under p is not contained in the image of K under p. The claim is that this weaker separation property is identical to the notion of rational convexity.

Indeed, if $p(w) \notin p(K)$, then for every sufficiently small positive ϵ, the function $1/(p(z) - p(w) - \epsilon)$ is a rational function of z that is holomorphic in a neighborhood of K and has larger modulus at w than it has anywhere on K. Conversely, if f is a rational function, holomorphic on $K \cup \{w\}$, whose modulus separates w from K, then the function $1/(f(z) - f(w))$ is a rational function of z that is holomorphic on K and undefined at w; if this function is rewritten as a quotient of polynomials, then the denominator is a polynomial that is equal to 0 at w and nonzero on K.

Thus, a point w is in the rationally convex hull of a compact set K if and only if every polynomial that is equal to zero at w also has a zero on K.

Example 8 *(the Hartogs triangle).* The open set $\{\, (z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1 \,\}$ is convex with respect to the linear fractional functions, because through each boundary point passes a complex line that does not intersect the domain. Indeed, the line $z_2 = 0$ serves at the origin $(0, 0)$; at any other boundary point where $|z_1| = |z_2|$, there is some value of θ for which the line $z_1 = e^{i\theta} z_2$ serves; and at a point where $|z_2| = 1$, there is some value of θ for which the line $e^{i\theta} z_2 = 1$ serves.

In particular, the open Hartogs triangle is a rationally convex domain, since there are more rational functions than there are linear fractions. On the other hand, the open Hartogs triangle is not polynomially convex. Indeed, the polynomial hull of the circle $\{\, (0, \frac{1}{2} e^{i\theta}) : 0 \le \theta < 2\pi \,\}$ is the disc bounded by this circle, but the center of the disc is not in the domain.

The situation changes if one considers the *closed* Hartogs triangle, the set where $|z_1| \le |z_2| \le 1$. The rationally convex hull of this compact set is the closed bidisc. Indeed, suppose p is a polynomial that has no zero on the closed Hartogs triangle; by continuity, p has no zero in an open neighborhood of the closed triangle. For each fixed z_1, the function $1/p$ has a Laurent series expansion in z_2 in the annulus where $|z_1| < |z_2| < 1$; when z_1 is close to 0, the series becomes a Maclaurin series that converges in a full disc where $|z_2| < 1$. The Laurent series coefficients are represented by integrals of the form

$$\frac{1}{2\pi i} \int_C \frac{1/p(z_1, w)}{w^{k+1}} \, dw,$$

so they depend holomorphically on z_1. Since the coefficients with negative index vanish for z_1 in a neighborhood of 0, they vanish identically. Hence the Laurent series for $1/p$ is a Maclaurin series even when z_1 is far away from 0. Consequently, the polynomial p cannot have any zeroes in the bidisc. By the characterization of rational convexity in terms of zeroes of polynomials, it follows that the rational hull of the closed Hartogs triangle contains the whole bidisc; the rational hull cannot contain any other points, since the rational hull is a subset of the convex hull.

The argument in the preceding paragraph is the same as the argument hinted at on page 1. The argument shows that a function holomorphic in an open neighborhood of the closed Hartogs triangle extends holomorphically to the whole bidisc. Thus the closed Hartogs triangle cannot be approximated from outside by polynomially convex domains or by rationally convex domains or even by holomorphically convex domains, which are the next topic to be discussed.

3.2.3 Holomorphic convexity

Suppose that G is a domain in $\mathbb{C}^n$, and $\mathcal{F}$ is the class of holomorphic functions on G. Then $\mathcal{F}$-convexity is called *holomorphic convexity* (with respect to G).

Example 9. When $G = \mathbb{C}^n$, holomorphic convexity is just polynomial convexity, since every entire function can be approximated uniformly on compact sets by polynomials (namely, the partial sums of the Maclaurin series).

If $G_1 \subset G_2$, and K is a compact subset of G_1, then the holomorphically convex hull of K with respect to G_1 evidently is a subset of the holomorphically convex hull of K with respect to G_2 (because there are more holomorphic functions on G_1 than there are on the larger domain G_2). In particular, a polynomially convex compact set is holomorphically convex with respect to any domain G that contains it; so is a convex set.

Example 10. Let K be the unit circle $\{ z \in \mathbb{C} : |z| = 1 \}$ in the complex plane.

(a) Suppose that G is the whole plane, in which case $\mathcal{F}$ is the class of entire functions. Then the $\mathcal{F}$-hull of K is the closed unit disc (by the maximum principle).

(b) Suppose that G is the punctured plane $\{\, z \in \mathbb{C} : z \neq 0 \,\}$, and $\mathcal{F}$ is the class of holomorphic functions on G. Then K is already an $\mathcal{F}$-convex set (because the function $1/z$, which is holomorphic on G, separates points inside the circle from points on the circle).

Thus the notion of holomorphic convexity does depend both on G and on K.

The next theorem solves the fundamental problem of characterizing the holomorphically convex domains in $\mathbb{C}^n$. This problem is interesting only when $n > 1$, for Example 7 implies that every domain in the complex plane is holomorphically convex. The theory of holomorphic convexity is due to Henri Cartan and Peter Thullen.[7]

Theorem 5. *The following properties are equivalent for a domain G in $\mathbb{C}^n$.*

1. *The domain G is holomorphically convex (that is, for every compact set K contained in G, the holomorphically convex hull $\widehat{K}$ is again a compact subset of G).*
2. *For every sequence of points in G having no accumulation point in G, there exists a holomorphic function on G that is unbounded on the sequence of points.*
3. *For every compact set K contained in G, the distance from K to the boundary of G is equal to the distance from $\widehat{K}$ to the boundary of G.*
4. *The domain G is a domain of holomorphy.*

The proof will include a precise definition of what property (4) means.

Proof. The initial steps in the proof will show that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1).

If the domain G is holomorphically convex, then G can be exhausted by an increasing sequence $\{K_j\}$ of compact, holomorphically convex sets. (Exhaust G by an arbitrary sequence of compact sets, and then replace each set by its holomorphically convex hull.) Now suppose given a sequence $\{p_j\}$ of points in G having no accumulation point in G. One can recursively construct a sequence of holomorphic functions f_k on G together with an increasing sequence of positive integers j_k such that $p_{j_k} \in K_{j_{k+1}} \setminus K_{j_k}$, and $|f_k(z)| < 2^{-k}$ for z in K_{j_k}, and $|f_k(p_{j_k})| > k + \sum_{m=1}^{k-1} |f_m(p_{j_k})|$. The functions f_k exist because K_{j_k} is holomorphically convex, and $p_{j_k} \notin K_{j_k}$. The series $\sum_{k=1}^{\infty} f_k$ then converges uniformly on compact subsets of G to a holomorphic function f such that $|f(p_{j_k})| > k-1$. Thus (1) $\Longrightarrow$ (2).

Next suppose that (2) holds, let K be a compact subset of G, and let w and t be points of $\widehat{K}$ and of bG at minimal distance from each other. Apply property (2) to produce a holomorphic function f on G that is unbounded on a sequence of points converging to t and lying in the set $\{\, w + \lambda(t-w)/|t-w| : \lambda \in \mathbb{C} \text{ and } |\lambda| < \operatorname{dist}(\widehat{K}, bG) \,\}$ (this set is a one-dimensional complex disc centered at w and lying in the complex line determined by w and t). The radius of convergence of the Maclaurin series expansion

[7] Henri Cartan and Peter Thullen, Zur Theorie der Singularitäten der Funktionen mehrer komplexen Veränderlichen. Regularitäts- und Konvergenzbereiche, *Mathematische Annalen* **106** (1932) 617–647.

of the function $\lambda \mapsto f(w + \lambda(t-w)/|t-w|)$ is then equal to the distance from $\widehat{K}$ to bG. The reciprocal of the radius of convergence is $\limsup_{m\to\infty} |a_m|^{1/m}$, where the Maclaurin series coefficients a_m, being given by derivatives of f, are the values at w of certain holomorphic functions on G. By the definition of the holomorphically convex hull, the reciprocal of the radius of convergence of the Maclaurin series for the function $\lambda \mapsto f(w + \lambda(t-w)/|t-w|)$ is no greater than the maximum as z varies over K of the reciprocal of the radius of convergence of the series representing the function $\lambda \mapsto f(z + \lambda(t-w)/|t-w|)$. Hence the radius of convergence is at least as large as the minimum over K of the radius of convergence of the series for $f(z + \lambda(t-w)/|t-w|)$, that is, at least as large as the minimum distance from K to bG along the direction of the vector $t-w$. In particular, the distance from $\widehat{K}$ to bG is at least as large as the distance from K to bG. On the other hand, $K \subseteq \widehat{K}$, so the distance from $\widehat{K}$ to bG is no less than the distance from K to bG. Thus (2) $\Longrightarrow$ (3).

If (3) holds, then $\widehat{K}$ stays away from bG. On the other hand, since K is bounded, so is its convex hull, and therefore so is its holomorphically convex hull (which is a subset of the convex hull). Hence $\widehat{K}$ is a compact subset of G. Thus (3) $\Longrightarrow$ (1).

Addressing property (4) requires an interruption of the proof to define precisely what a domain of holomorphy is. □

According to Cartan and Thullen, a domain is a domain of holomorphy if it supports a holomorphic function that does not extend holomorphically to any larger domain,[8] but to Cartan and Thullen, a "domain" is a Riemann domain spread over $\mathbb{C}^n$ (the higher-dimensional analogue of a Riemann surface), that is, a complex manifold. The definition below is (necessarily) slightly convoluted because it formulates the concept of domain of holomorphy without introducing the machinery of manifolds.

Example 11. One can define a holomorphic branch of the function $\sqrt{z}$ on the slit plane $\mathbb{C} \setminus \{\, z : \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \le 0 \,\}$. This function is discontinuous at all points of the negative part of the real axis, so the function certainly does not extend to be holomorphic in a neighborhood of any of these points. The function does, however, continue holomorphically across each non-zero boundary point *from one side.* The natural domain of definition of $\sqrt{z}$ is not the slit plane but rather a two-sheeted Riemann surface.

In general, the boundary of a domain can be quite complicated. For instance, the boundary need not be locally connected. A holomorphic function f on a domain G is called *completely singular* at a boundary point p if for every connected open neighborhood U of p, there does *not* exist a holomorphic function F on U that agrees with f on some nonvoid open subset of $U \cap G$ (equivalently, on some connected component of $U \cap G$). A completely singular function is "holomorphically non-extendable" in the strongest possible way.

A domain G in $\mathbb{C}^n$ is called a *domain of holomorphy* if there exists a holomorphic function on G that is completely singular at every boundary point of G. This property

[8] "Einen Bereich $\mathfrak{B}$ nennen wir einen *Regularitätsbereich (domaine d'holomorphie)*, falls es eine in $\mathfrak{B}$ eindeutige und reguläre Funktion $f(z_1, \ldots, z_n)$ gibt derart, daß jeder $\mathfrak{B}$ enthaltende Bereich $\mathfrak{B}'$, in dem $f(z_1, \ldots, z_n)$ eindeutig und regulär ist, notwendig mit $\mathfrak{B}$ identisch ist." Cartan and Thullen, loc. cit., p. 618.

appears to be hard to verify in concrete cases. An easier property to check is the existence at each boundary point p of a holomorphic function on G that is completely singular at p; a domain satisfying this (apparently less restrictive) property is sometimes called a *weak domain of holomorphy*. The next theorem shows that a weak domain of holomorphy is in fact a domain of holomorphy.

Example 12. Convex domains are weak domains of holomorphy. Indeed, at each boundary point there is an affine complex linear function that is zero at the boundary point but nonzero inside the domain. The reciprocal of the function is then holomorphic inside and singular at the specified boundary point. It is less obvious how to exhibit a holomorphic function that is singular at every boundary point of a convex domain.

The next theorem subsumes the remaining parts of Theorem 5 and also shows that weak domains of holomorphy are domains of holomorphy.

Theorem 6. *The following properties of a domain G in $\mathbb{C}^n$ are equivalent.*

1. *The domain G is holomorphically convex.*
2. *For every boundary point p, there exists a holomorphic function on G that is completely singular at p. (In other words, G is a weak domain of holomorphy.)*
3. *For every boundary point p, the generic (in the sense of Baire category) holomorphic function on G is completely singular at p.*
4. *There exists a holomorphic function on G that is completely singular at every boundary point of G. (In other words, G is a domain of holomorphy.)*
5. *The generic (in the sense of Baire category) holomorphic function on G is completely singular at every boundary point.*

The space of holomorphic functions on a domain G carries a topology induced by uniform convergence on compact subsets (normal convergence), and this topology is metrizable. Namely, exhaust G by an increasing sequence of compact subsets K_j, let $d_j(f,g) = \max_{z\in K_j} |f(z) - g(z)|$, and define a metric on holomorphic functions by $d(f,g) = \sum_j 2^{-j} d_j(f,g)/(1 + d_j(f,g))$. Since the normal limit of holomorphic functions is holomorphic, the space of holomorphic functions on G is a complete metric space. This property was already used on page 8 in the proof of Theorem 2 to bring in the Baire category theorem. The word "generic" in the statement of Theorem 6 means "the complement of a set of first Baire category".

Proof of Theorem 6. Suppose that G is a holomorphically convex domain. To get started constructing singular functions, let p be a boundary point of G, let U be a connected open neighborhood of p, and let V be a connected component of the intersection $U \cap G$.

The first observation is a purely topological one: namely, there exists a point q in $(U \cap bV) \cap bG$. Indeed, since V is a proper open subset of U, and U is connected, it follows that V cannot be relatively closed in U. Hence some point of bV lies in $U \setminus V$. If this point of $U \cap bV$ were in G, then it would in particular be in $U \cap G$ and hence in some connected component of $U \cap G$ other than V; this component would intersect bV and hence V, which is impossible.

By Theorem 5, there exists a holomorphic function on G that is unbounded on a sequence of points of V tending to q. Since $q \in U$, this function cannot be extended holomorphically from V to U.

The next claim is that *most* holomorphic functions on G cannot be extended holomorphically from V to U. The vector space of holomorphic functions on G is not only a complete metric space but also an F-space or Fréchet space (that is, the vector space operations are continuous, and the metric is translation-invariant); a standard notation for this space is $\mathcal{O}(G)$. The subspace of $\mathcal{O}(G)$ consisting of functions that extend holomorphically from V to U can be viewed as a Fréchet space whose metric is the sum of the metrics from $\mathcal{O}(G)$ and $\mathcal{O}(U)$; this subspace is embedded continuously into $\mathcal{O}(G)$. As just shown, the image of the embedding is not the whole of $\mathcal{O}(G)$, so by a theorem from functional analysis, the image is of first Baire category.[9] Thus a residual set of functions in $\mathcal{O}(G)$ cannot be extended holomorphically from V to U.

To strengthen the conclusion further, choose a countable dense set of points in bG. For each point, choose a countable neighborhood basis of open balls centered at the point, say the balls having radius $1/k$ as k runs through the positive integers. The intersection of each ball with G has either a finite or a countably infinite number of connected components. Arrange the collection of components over all balls and all points into a countable list $\{V_j\}_{j=1}^{\infty}$. According to what was just shown, the set of holomorphic functions on G that extend holomorphically from a particular V_j to its corresponding ball is a set of first category in $\mathcal{O}(G)$. Therefore the set of holomorphic functions on G that extend from any V_j at all is a countable union of sets of first category, hence still a set of first category. In other words, the complementary set of holomorphic functions on G that extend from no V_j to the corresponding ball is a residual set.

It seems plausible that every member of this residual set of holomorphic functions must be completely singular at every boundary point. One can confirm this expectation as follows. Suppose that some function f in the residual set fails to be completely singular at a boundary point p. This means that there is a connected neighborhood U of p and a component V of $U \cap G$ such that f extends holomorphically from V to U. As shown earlier in the proof, there is some point q in the intersection $(U \cap bV) \cap bG$. Choose one of the specified countable set of open balls that contains q and is contained in U. This ball intersects V (since it contains a point of bV), so the function f extends holomorphically from an open subset of the ball to the whole ball (a subset of U). By construction, however, the function f does not admit such an extension. The contradiction shows that every function f in the indicated residual set of functions is completely singular at every boundary point of G.

In summary, the preceding argument shows that if property (1) holds, then property (5) holds. It is evident that (5) $\Longrightarrow$ (4) $\Longrightarrow$ (2) and (5) $\Longrightarrow$ (3) $\Longrightarrow$ (2).

[9] The argument is the same as the one on page 9. The theorem from Banach's book cited there applies, or one could invoke the version of the open mapping theorem from Walter Rudin's book *Functional Analysis* (section 2.11, page 48 of the second edition): a continuous linear mapping between Fréchet spaces either is a surjective open map or has image of first category.

It remains to show that (2) $\Longrightarrow$ (1), in other words, that a weak domain of holomorphy is holomorphically convex. The strategy is to show that $\operatorname{dist}(K, bG) = \operatorname{dist}(\widehat{K}, bG)$ for every compact subset K of G by using a similar approach to that in the proof of Theorem 5. It is enough to show that $\operatorname{dist}(\widehat{K}, bG) \geq \operatorname{dist}(K, bG)$, for the reverse inequality follows simply because $K \subseteq \widehat{K}$.

Seeking a contradiction, suppose that $\operatorname{dist}(\widehat{K}, bG) < \operatorname{dist}(K, bG)$. There is then a point w in $\widehat{K}$ and a point p in bG such that $|w - p| < \operatorname{dist}(K, bG)$. It follows that there is an n-tuple $(r_1, \ldots, r_n)$ of positive radii such that the open polydisc centered at w with polyradius r equal to $(r_1, \ldots, r_n)$ contains p, while for every point z in K, the closed polydisc centered at z with polyradius r is contained in G.

Under the hypothesis that G is a weak domain of holomorphy, there is a holomorphic function f on G that is completely singular at p. The union of the closed polydiscs centered at points of the compact set K with polyradius r is a compact subset of G, so the function f is bounded by some constant M on this set. By Cauchy's estimates for derivatives (just as in one variable, these inequalities follow from the iterated Cauchy integral on polydiscs),

$$|f^{(\alpha)}(z)| \leq \frac{M\alpha!}{r^{\alpha}} \qquad \text{for } z \text{ in } K \text{ and for every multi-index } \alpha.$$

Since $w \in \widehat{K}$, the same inequalities hold with z replaced by w. Consequently, the Taylor series for f centered at w converges in the interior of the polydisc centered at w with polyradius r (by comparison with a product of convergent geometric series).

Thus f is not completely singular at p after all. The contradiction shows that the supposition that $\operatorname{dist}(\widehat{K}, bG) < \operatorname{dist}(K, bG)$ is untenable.

This conclusion proves that (2) implies (1), which completes the chain of implications required to prove the theorem. □

Exercise 12. For each of the following subsets of $\mathbb{C}^2$, determine if the subset is a domain of holomorphy.

(a) The complement of a point.

(b) The complement of the real line $\{\, (z_1, 0) : \operatorname{Im} z_1 = 0 \,\}$.

(c) The complement of the complex line $\{\, (z_1, 0) : z_1 \in \mathbb{C} \,\}$.

(d) The complement of the totally real 2-plane $\{\, (z_1, z_2) : \text{both } \operatorname{Im} z_1 = 0 \text{ and } \operatorname{Im} z_2 = 0 \,\}$.

(e) The complement of the half-line $\{\, (z_1, 0) : \operatorname{Im} z_1 \geq 0 \,\}$.

(f) The complement of $\{\, (z_1, 0) : z_1 \in \mathbb{C} \,\} \cup \{\, (0, z_2) : z_2 \in \mathbb{C} \,\} \cup \{\, (z_1, z_2) : z_1 \neq 0$ and $z_2 \neq 0$ and $\arg z_1 = \arg z_2 \,\}$. The removed set consists of two complex lines together with a certain surface. If you are worried about the argument function not being well defined, then rewrite the condition $\arg z_1 = \arg z_2$ as $z_1/|z_1| = z_2/|z_2|$.

Exercise 13. (a) Is the union of two domains of holomorphy again a domain of holomorphy?

(b) If the intersection of two domains of holomorphy is a nonvoid connected set, is it again a domain of holomorphy?

(c) If G_1 is a domain of holomorphy in $\mathbb{C}^{n_1}$, and G_2 is a domain of holomorphy in $\mathbb{C}^{n_2}$, is the Cartesian product $G_1 \times G_2$ a domain of holomorphy in $C^{n_1+n_2}$?

(d) Show that holomorphic convexity is a biholomorphically invariant property: namely, if $f\colon G_1 \to G_2$ is a bijective holomorphic map having a holomorphic inverse, then G_1 is a domain of holomorphy if and only if G_2 is a domain of holomorphy.

(e) Suppose G is a domain of holomorphy in $\mathbb{C}^n$, and $f\colon G \to \mathbb{C}^n$ is a holomorphic map (not necessarily either injective or surjective). If the image $f(G)$ is an open set in $\mathbb{C}^n$, is it a domain of holomorphy?

(f) Suppose G is a domain of holomorphy in $\mathbb{C}^n$, and $f\colon G \to \mathbb{C}^k$ is a holomorphic map (not necessarily either injective or surjective). Show that if D is a domain of holomorphy in $\mathbb{C}^k$, then (each connected component of) the inverse image $f^{-1}(D)$ [that is, $\{\, z \in G : f(z) \in D \,\}$] is a domain of holomorphy in $\mathbb{C}^n$.

(g) If $f_1, \ldots, f_k$ are holomorphic functions on a holomorphically convex domain, then each connected component of $\{\, z : \sum_{j=1}^{k} |f_j(z)| < 1 \,\}$ is a domain of holomorphy.

3.2.4 Pseudoconvexity

Pseudoconvexity means convexity with respect to a certain class of real-valued functions that Kiyoshi Oka[10] called "pseudoconvex functions". Pierre Lelong[11] called these functions "plurisubharmonic functions", and this is the name by which they are now known. The discussion had better start with the base case of dimension 1.

Subharmonic functions

A real-valued function u defined on an open subset of the complex plane $\mathbb{C}$ and taking values in $[-\infty, \infty)$ is called *subharmonic* if firstly it is upper semi-continuous, and secondly it satisfies one of the following equivalent properties:

1. for every point a in the domain of u, there is a radius $r(a)$ such that u satisfies the sub-mean-value property on every disc of radius ρ less than $r(a)$, that is, $u(a) \le \frac{1}{2\pi}\int_0^{2\pi} u(a+\rho e^{i\theta})\,d\theta$;
2. the function u satisfies the sub-mean-value property on every closed disc contained in its domain;

[10]Kiyoshi Oka, Domaines pseudoconvexes, *Tôhoku Mathematical Journal* **49** (1942) 15–52.

[11]Pierre Lelong, Définition des fonctions plurisousharmoniques, *C. R. Acad. Sci. Paris* **215** (1942) 398–400.

3. for every closed disc D in the domain of the function u and every harmonic function h on D, if $u \le h$ on the boundary of D, then $u \le h$ in all of D;

4. for every compact subset K of the domain of definition of u, and for every function h that is harmonic on K, if $u \le h$ on the boundary of K, then $u \le h$ on all of K;

5. if Δ denotes the Laplace operator $\frac{d^2}{dx^2} + \frac{d^2}{dy^2}$, then $\Delta u \ge 0$ (if u does not have second derivatives in the classical sense, then Δu is understood in the sense of distributions).

That these properties are equivalent is shown in textbooks on the theory of functions of one complex variable. Some authors exclude the function that is constantly equal to $-\infty$ from the class of subharmonic functions.

A simple example of a subharmonic function is $|f|$, where f is holomorphic. Since a holomorphic function has the mean-value property, its modulus has the sub-mean-value property because the modulus of an integral does not exceed the integral of the modulus.

Another basic example of a subharmonic function in $\mathbb{C}$ is $\log|z|$. This function is even harmonic when $z \ne 0$, so it has the mean-value property on small discs centered at non-zero points; it trivially has the sub-mean-value property at 0, because it takes the value $-\infty$ at 0. Since the class of harmonic functions is preserved under composition with a holomorphic function, property 4 implies that the class of subharmonic functions is preserved too: so $\log|f|$ is subharmonic when f is holomorphic.

Here are two useful lemmas about subharmonic functions that can be proved from first principles.

Lemma 2. If u is subharmonic, then the integral of u on concentric circles is a weakly increasing function of the radius. In other words, $\int_0^{2\pi} u(a+r_1e^{i\theta})\,d\theta \le \int_0^{2\pi} u(a+r_2e^{i\theta})\,d\theta$ when $0 < r_1 < r_2$.

Lemma 3. A subharmonic function on a connected open set is either locally integrable or identically equal to $-\infty$.

Proof of Lemma 2. Since u is upper semi-continuous, there is for each positive ε a continuous function h on the circle of radius r_2 such that $u < h < u + \varepsilon$ on this circle. By solving a Dirichlet problem, one may assume that h is harmonic in the disc of radius r_2, or, after slightly dilating the coordinates, in a neighborhood of the closed disc. Then $u < h$ on the circle of radius r_1, since u is subharmonic, so $\int_0^{2\pi} u(a + r_1e^{i\theta})\,d\theta < \int_0^{2\pi} h(a + r_1e^{i\theta})\,d\theta = 2\pi h(a) = \int_0^{2\pi} h(a + r_2e^{i\theta})\,d\theta < 2\pi\varepsilon + \int_0^{2\pi} u(a + r_2e^{i\theta})\,d\theta$. Letting ε go to 0 gives the required inequality. □

Proof of Lemma 3. An upper semi-continuous function is locally bounded above, so what needs to be proved is that the integral of the subharmonic function u on a disc is not $-\infty$ unless the function is identically equal to $-\infty$.

If a is a point at which $u(a) \ne -\infty$, then the sub-mean-value property implies that $\frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\,d\theta \ge u(a)$ when the closed disc centered at a of radius r is contained in the domain of u. Averaging in r shows that $|B|^{-1}\int_B u \ge u(a)$ for every ball B centered at a. Hence u is locally integrable in a neighborhood of every point of B.

On the other hand, if b is a point such that $u(b) = -\infty$, but u is not identically equal to $-\infty$ in a neighborhood of b, then there is a point a closer to b than to the boundary of the domain of definition of u such that $u(a) \neq -\infty$. Then by what was just observed, the function u is integrable in a neighborhood of b.

The preceding two paragraphs show that the set of points such that u is integrable in a neighborhood of the point is both open and relatively closed. Therefore the function u, if not identically equal to $-\infty$, is locally integrable in a neighborhood of every point of its domain.

□

Exercise 14. (a) The sum of two subharmonic functions is subharmonic.

(b) If u is subharmonic and c is a positive constant, then cu is subharmonic.

(c) If u_1 and u_2 are subharmonic, then so is the pointwise maximum of u_1 and u_2.

Some care is needed in handling infinite processes involving subharmonic functions. Indeed, simple examples show that two things could go wrong in taking the pointwise supremum of an infinite family of subharmonic functions. The sequence of constant subharmonic functions $f_n(z) = n$ has limit $+\infty$, which is not an allowed value for upper semi-continuous functions. On the unit disc, the family of subharmonic functions $f_n(z) = \frac{1}{n}\log|z|$ has pointwise supremum equal to 0 when $z \neq 0$ and equal to $-\infty$ when $z = 0$; this limit function is not upper semi-continuous. The following exercise says that these difficulties are the only obstructions to subharmonicity of a pointwise supremum.

Exercise 15. If A is any index set (not necessarily countable), u_α is subharmonic for each α in A, and the pointwise supremum $\sup_{\alpha\in A} u_\alpha$ is upper semi-continuous (which entails being nowhere equal to $+\infty$), then the pointwise supremum is subharmonic.

Taking a pointwise supremum of subharmonic functions is a process used in Perron's method for solving the Dirichlet problem.

Although taking the maximum of two subharmonic functions produces another one, taking the minimum does not. For instance, $\min(1, |z|)$ does not have the sub-mean-value property at the point where $z = 1$. Nonetheless, monotonically decreasing sequences of subharmonic functions have subharmonic limits.

Theorem 7. *The pointwise limit of a decreasing sequence of subharmonic functions is subharmonic. Moreover, every subharmonic function on an open set is, on each compact subset, the limit of a decreasing sequence of infinitely differentiable subharmonic functions.*

Proof. First observe that the limit u of a decreasing sequence $\{u_k\}_{k=1}^\infty$ of upper semi-continuous functions is still upper semi-continuous, because $\{\, z : u(z) < a \,\} = \bigcup_{k=1}^\infty \{\, z : u_k(z) < a \,\}$, and the union of open sets is open. Now if K is a compact subset of the domain of definition of the functions, and h is a harmonic function on K such that $u \le h$ on the boundary of K, then $u < h + \varepsilon$ on the boundary of K for every positive ε. If

z is a point of bK, then $u_k(z) < h(z) + \varepsilon$ for all sufficiently large k. Since u_k is upper semi-continuous, if follows that $u_k(w) \le h(w) + 2\varepsilon$ for all w in a neighborhood of z. Since bK is compact, and the sequence of functions is decreasing, there is some k such that $u_k \le h + 2\varepsilon$ on all of bK. Since u_k is subharmonic, $u_k \le h + 2\varepsilon$ on all of K. Therefore $u \le h + 2\varepsilon$ on K, and letting ε go to 0 shows that $u \le h$ on K. Hence the limit function u is subharmonic.

For the second part of the theorem, let u be a subharmonic function on a domain G in $\mathbb{C}$, and extend u to be identically equal to 0 outside G. Let φ be an infinitely differentiable, non-negative function, with integral 1, supported in the unit ball, and depending only on the radius, and let $\varphi_\varepsilon(x)$ denote $\varepsilon^{-2}\varphi(x/\varepsilon)$. Let u_ε denote the convolution of u and φ_ε: namely, $u_\varepsilon(z) = \int_{\mathbb{C}} \varphi_\varepsilon(z-w)u(w)\,dA_w = \int_{\mathbb{C}} u(z-w)\varphi_\varepsilon(w)\,dA_w$, where dA denotes Lebesgue area measure in the plane. Thus the value of u_ε at a point is a weighted average of the values of u in an ε-neighborhood of the point.

The sub-mean-value property of subharmonic functions implies that $u(z) \le u_\varepsilon(z)$ at every point z whose distance from the boundary of G is at least ε. Moreover, Lemma 2 implies that on a compact subset of G, the functions u_ε decrease when ε decreases, once ε is smaller than the distance from the compact set to bG. Since u is upper semi-continuous, the average of u over a sufficiently small disc is arbitrarily little more than the value of u at the center of the disc; the decreasing limit of $u_\varepsilon(z)$ is therefore equal to $u(z)$. The first expression for the convolution shows that the functions u_ε are infinitely differentiable, for one can differentiate under the integral sign, letting the derivatives act on φ_ε. That u_ε is subharmonic follows by integrating u_ε on a circle, interchanging the order of integration, and invoking the subharmonicity of u. $\square$

Here are two interesting examples that apply Theorem 7.

Example 13. Let $\{a_k\}_{k=1}^{\infty}$ be a bounded sequence of distinct points of the plane $\mathbb{C}$, and suppose $u(z) = \sum_{k=1}^{\infty} 2^{-k} \log|z - a_k|$. Then u is a subharmonic function on the whole plane. Notice that the sequence $\{a_k\}$ might be dense in some compact set; for instance, the sequence could be the set of points in the unit square having rational coordinates.

To see why u is subharmonic, first suppose that z_0 is not one of the points a_k nor a limit point of the sequence. Then $\log|z - a_k|$ is bounded above and below in a neighborhood of z_0, so the series defining $u(z)$ converges uniformly in the neighborhood. The limit of a uniformly convergent series of harmonic functions is harmonic, so $u(z)$ is harmonic off the closure of the sequence $\{a_k\}$.

Next suppose that z_0 is a point in the closure of the sequence $\{a_k\}$. Split the sum defining $u(z)$ into the sum of terms for which $|a_k - z_0| < 1/2$ and the sum of terms for which $|a_k - z_0| \ge 1/2$. The second sum converges uniformly for z in a neighborhood of z_0 (as in the preceding paragraph) and represents a harmonic function there. The first sum is a sum of negative terms (for z in a neighborhood of z_0), so the partial sums form a decreasing sequence of subharmonic functions. By Theorem 7, the partial sums converge to a subharmonic function.

Thus u is subharmonic in the whole plane $\mathbb{C}$. Notice that u takes the value $-\infty$ at each point a_k, but the set where u equals $-\infty$ is a set of measure zero, since the subharmonic function u is locally integrable by Lemma 3.

Example 14. Let G be a proper subdomain of the complex plane. Then $-\log \operatorname{dist}(z, bG)$ is a subharmonic function for z in G.

Indeed, if a is a point of bG, then $-\log|z-a|$ is a harmonic function on G. Since

$$\sup\{\,-\log|z-a| : a \in bG\,\} = -\inf\{\,\log|z-a| : a \in bG\,\} = -\log \operatorname{dist}(z, bG),$$

it is a consequence of Theorem 7 that $-\log \operatorname{dist}(z, bG)$ is subharmonic.

Plurisubharmonic functions

Introduction An upper semi-continuous function is called a *plurisubharmonic* function if its restriction to every complex line is subharmonic. The name and the fundamental properties of plurisubharmonic functions are due to Lelong.[12]

It will turn out that a proper subdomain of $\mathbb{C}^n$ is convex with respect to the plurisubharmonic functions if and only if $-\log \operatorname{dist}(z, bG)$ is a plurisubharmonic function for z in G; a domain satisfying this property is called *pseudoconvex*.

Since $\log|f|$ is plurisubharmonic when f is holomorphic, it follows that every holomorphically convex domain is pseudoconvex. The famous *Levi problem*, to be solved later, is to prove the converse: every pseudoconvex domain is a domain of holomorphy.

Equivalent definitions Suppose that u is an upper semi-continuous function on a domain D in $\mathbb{C}^n$. The following properties are all equivalent to u being a plurisubharmonic function on D.

1. For every point z in D and every vector w in $\mathbb{C}^n$, the function $\lambda \mapsto u(z + \lambda w)$ is a subharmonic function of λ in $\mathbb{C}$ where it is defined. (This is the precise statement of what it means for the restriction of u to every complex line to be subharmonic.)

2. For every holomorphic mapping from the unit disc into D, the composite function $u \circ f$ is subharmonic on the unit disc. (In other words, the restriction of u to a one-dimensional complex variety is subharmonic.)

3. If u is twice continuously differentiable, then

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \geq 0 \qquad \text{for every vector } w \text{ in } \mathbb{C}^n.$$

   The notation $\partial/\partial z_j$ means $\frac{1}{2}(\partial/\partial x_j - i\partial/\partial y_j)$ in terms of the underlying real coordinates for which $z_j = x_j + iy_j$. Similarly, $\partial/\partial \overline{z}_j$ means $\frac{1}{2}(\partial/\partial x_j + i\partial/\partial y_j)$. This notation for complex partial derivatives seems to be due to the Austrian mathematician Wilhelm Wirtinger.[13]

[12] Pierre Lelong, Les fonctions plurisousharmoniques, *Annales scientifiques de l'École Normale Supérieure Sér. 3* **62** (1945) 301–338.

[13] W. Wirtinger, Zur formalen Theorie der Funktionen von mehr komplexen Veränderlichen, *Mathematische Annalen* **97** (1927) 357–376.

If u is not twice differentiable, then one can interpret the preceding inequality in the sense of distributions. Alternatively, u is the limit of a decreasing sequence of infinitely differentiable functions satisfying the inequality.

4. For every closed polydisc of arbitrary orientation contained in D, the value of u at the center of the polydisc is at most the average of u on the torus in the boundary of the polydisc. It is equivalent to say that each point of D has a neighborhood such that the indicated property holds for polydiscs contained in the neighborhood.

The last property needs some explanation. A polydisc is a product of one-dimensional discs. Part of the boundary of the polydisc is the Cartesian product of the boundaries of the one-dimensional discs. This Cartesian product of circles is a multi-dimensional torus. There is no standard designation for this torus, which different authors call by various names, such as "distinguished boundary", "skeleton", and "spine". Lelong uses the French word "arète". The words "arbitrary orientation" mean that the polydisc need not have its sides parallel to the coordinate axes: the polydisc could be rotated by a unitary transformation.

It is useful to look at some examples of plurisubharmonic functions before proving the equivalence of the various properties. If f is a holomorphic function, then $|f|$ and $\log|f|$ are plurisubharmonic because the restriction of f to every complex line is a holomorphic function of one variable.

Less obvious examples are $\log(|z_1|^2 + |z_2|^2)$ and $\log(1 + |z_1|^2 + |z_2|^2)$. The plurisubharmonicity could be verified by computing second derivatives and checking that the complex Hessian matrix is non-negative, but here is an alternate approach that handles the higher-dimensional analogue with no extra work. Observe that $|z_1|^2 + |z_2|^2 = \sup\{\, |z_1\overline{w}_1 + z_2\overline{w}_2| : |w_1|^2 + |w_2|^2 = 1 \,\}$. Now for fixed values of w_1 and w_2, the function $z_1\overline{w}_1 + z_2\overline{w}_2$ is a holomorphic function of z_1 and z_2, so $\log|z_1\overline{w}_1 + z_2\overline{w}_2|$ is plurisubharmonic. The supremum of a family of plurisubharmonic functions, if upper semicontinuous, is plurisubharmonic [just as in the one-dimensional case], so $\log(|z_1|^2+|z_2|^2)$ is plurisubharmonic. The same argument shows that $\log(|z_1|^2+|z_2|^2+|z_3|^2)$ is a plurisubharmonic function in $\mathbb{C}^3$, and fixing z_3 equal to 1 shows that $\log(1 + |z_1|^2 + |z_2|^2)$ is a plurisubharmonic function in $\mathbb{C}^2$.

The example $(\log|z_1|)(\log|z_2|)$ shows that a function can be subharmonic in each variable separately without being plurisubharmonic. On the open set where $z_1z_2 \neq 0$, the function is even harmonic in each variable separately, but the determinant of the complex Hessian is negative, so the function is not plurisubharmonic. A function that is subharmonic in each variable separately has the sub-mean-value property on polydiscs with faces parallel to the coordinate axes, so property (4) needs to allow polydiscs of arbitrary orientation.

It is tempting to try to extend the list of equivalent properties in parallel with the equivalent properties for subharmonicity. One might define a function to be "subpluriharmonic" if whenever it is bounded above on the boundary of a compact set by a pluriharmonic function, then it is bounded above on the whole set by the pluriharmonic function. (A function is pluriharmonic if its restriction to each complex line is harmonic. Equivalently, a function is pluriharmonic if locally it is the real part of a holomorphic function.) A plurisubharmonic function is subpluriharmonic, but the converse is false. Indeed, every plurisubharmonic function is subharmonic (as a function on $\mathbb{R}^{2n}$), every pluriharmonic function is harmonic, and every subharmonic function is subpluriharmonic. The function $|z_1|^2 - |z_2|^2$ is harmonic (as a function on $\mathbb{R}^4$), hence subpluriharmonic, but not plurisubharmonic.

Proof of the equivalence. Suppose at first that u is twice continuously differentiable. In that case, saying that $u(z + \lambda w)$ is subharmonic as a function of λ is the same as saying that the Laplacian is non-negative. The Laplacian is $4 \sum_{j=1}^{n} \sum_{k=1}^{n} u_{j\overline{k}} w_j \overline{w}_k$. Hence property (1) implies property (3).

To see that (3) implies (2), observe that the composite function $u \circ f$ is subharmonic precisely when its Laplacian is non-negative, and when f is a holomorphic mapping, the chain rule implies that the Laplacian of $u \circ f$ equals $4 \sum_{j=1}^{n} \sum_{k=1}^{n} u_{j\overline{k}} f_j' \overline{f_k'}$.

Evidently property (2) implies property (1). Hence properties (1), (2), and (3) are all equivalent when u is smooth.

When u is not smooth, property (2) still trivially implies property (1). To prove the converse, take a decreasing sequence of smooth plurisubharmonic functions converging to u (by convolving u with smooth mollifying functions, just as in one variable). For the smooth approximants, property (2) holds by the first part of the proof, and this property evidently continues to hold in the limit.

It remains to show that property (1) is equivalent to property (4). Observe that the average of an upper semi-continuous function over a sufficiently small circle exceeds the value of the function at the center of the circle by arbitrarily little. If (4) holds, then let $n - 1$ of the radii tend to 0 to see that the restriction of u to a disc in a complex line satisfies the sub-mean-value property. Thus (4) implies (1). Conversely, suppose (1) holds. Now property (1) is unchanged by composition with a unitary transformation (since unitary transformations take complex lines to complex lines), so it suffices to check (4) for polydiscs with their faces parallel to the coordinate planes. Integrate on the torus by integrating over each circle separately. Applying (1) for each integral shows that (4) holds. □

Before stating a theorem characterizing pseudoconvexity, it is convenient to formulate yet another property, the continuity principle (also known as the *Kontinuitätssatz*). An *analytic disc* is a continuous mapping from the closed unit disc D in $\mathbb{C}$ into $\mathbb{C}^n$ that is holomorphic on the open disc. Often an analytic disc is identified with its image (at least when the mapping is one-to-one). Here are two versions of the continuity principle for analytic discs. The principle may or may not hold for a particular domain G in $\mathbb{C}^n$.

(a) If for each α in some index set A, the mapping $f_\alpha \colon D \to G$ is an analytic disc whose image is contained in the domain G, and if there is a compact subset of G that contains $\bigcup_{\alpha \in A} f_\alpha(bD)$ (the "boundaries" of the analytic discs), then there is a compact subset of G that contains $\bigcup_{\alpha \in A} f_\alpha(D)$.

(b) If $f_t \colon D \to \mathbb{C}^n$ is a family of analytic discs varying continuously with respect to the parameter t in the interval $[0, 1]$, if $\bigcup_{0 \le t \le 1} f_t(bD)$ is contained in the domain G

(hence automatically contained in a compact subset of G), and if $f_0(D)$ is contained in G, then $\bigcup_{0\le t\le 1} f_t(D)$ is contained in G (hence in a compact subset of G).

Here is the statement of the theorem that characterizes pseudoconvex domains by four equivalent properties. A sufficiently smooth function is called *strictly* (or *strongly*) plurisubharmonic if its complex Hessian matrix is positive definite (rather than semi-definite).

Theorem 8. *The following properties of a domain G in $\mathbb{C}^n$ are equivalent.*

1. *There exists an infinitely differentiable, strictly plurisubharmonic exhaustion function for G.*
2. *The domain G is convex with respect to the plurisubharmonic functions.*
3. *The continuity principle holds for G.*
4. *The function $-\log \operatorname{dist}(z, bG)$ is plurisubharmonic.*

Exercise 16. The unit ball $\{\, z \in \mathbb{C}^n : \|z\| < 1 \,\}$ (where $\|z\|^2 = |z_1|^2 + \cdots + \|z_n|^2$) is convex, hence convex with respect to the holomorphic functions, hence convex with respect to the plurisubharmonic functions. The distance from z to the boundary equals $1 - \|z\|$. Verify that $-\log(1 - \|z\|)$ is plurisubharmonic and that $-\log(1 - \|z\|^2)$ is an infinitely differentiable, plurisubharmonic exhaustion function.

Proof of Theorem 8. The plan of the proof is (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1).

It is easy to see that (1) implies (2): if K is a compact subset of G, and u is a plurisubharmonic exhaustion function, then u is bounded above on K by some constant M, and the plurisubharmonic hull of K is contained in $\{\, z \in G : u(z) \le M \,\}$, which by is a compact subset of G by the definition of an exhaustion function.

Suppose (2) holds. If $f\colon D \to G$ is an analytic disc, and u is a plurisubharmonic function on G, then $u \circ f$ is a subharmonic function on the unit disc, so $u(f(t)) \le \max\{\, u(f(e^{i\theta})) : 0 \le \theta \le 2\pi \,\}$ for every point t in D. In other words, $f(D)$ is contained in the plurisubharmonic hull of $f(bD)$. Hence version (a) of the continuity principle holds: the plurisubharmonic hull of the compact set containing $\bigcup_{\alpha\in A} f_\alpha(bD)$ is a compact set containing $\bigcup_{\alpha\in A} f_\alpha(D)$. To get version (b) of the continuity principle, consider the set S of points t in the interval $[0, 1]$ for which $f_t(D) \subset G$. This set is nonvoid, since it contains 0 by hypothesis. If $t \in S$, then $f_t(D)$ is a compact subset of G (since f_t is continuous on the closed disc D), so $f_s(D) \subset G$ for s near t (since the discs vary continuously with respect to the parameter). Thus S is a closed set. Version (a) of the continuity principle implies that the set S is closed. Hence S is all of $[0, 1]$, which is what needed to be shown. Consequently, (2) implies (3).

Suppose that (3) holds. To see that $-\log \operatorname{dist}(z, bG)$ is plurisubharmonic, fix a point z_0 in G and a vector w_0 in $\mathbb{C}^n$ such that the closed disc $\{\, z_0 + \lambda w_0 : |\lambda| \le 1 \,\}$ lies in G. To show that $-\log \operatorname{dist}(z_0 + \lambda w_0, bG)$ is subharmonic as a function of λ, it suffices to fix a polynomial p of one complex variable such that

$$-\log \operatorname{dist}(z_0 + \lambda w_0, bG) \le \operatorname{Re} p(\lambda) \qquad \text{when } |\lambda| = 1$$

and to show that the same inequality holds when $|\lambda| < 1$. This problem translates directly into the equivalent problem of showing that if

$$\operatorname{dist}(z_0 + \lambda w_0, bG) \geq |e^{-p(\lambda)}| \qquad \text{when } |\lambda| = 1,$$

then the same inequality holds when $|\lambda| < 1$. A further reformulation is to show that if, for every point ζ in the open unit ball of $\mathbb{C}^n$, the point $z_0 + \lambda w_0 + \zeta e^{-p(\lambda)}$ lies in G when $|\lambda| = 1$, then the same property holds when $|\lambda| < 1$.

Now $\lambda \mapsto z_0 + \lambda w_0 + \zeta e^{-p(\lambda)}$ is a continuous family of analytic discs in $\mathbb{C}^n$ parametrized by ζ. When $\zeta = 0$, the analytic disc lies in G by hypothesis. Also by hypothesis, the boundaries of the analytic discs lie in G. Hence version (b) of the continuity principle (applied to the line segment joining 0 to ζ) implies that the analytic discs all lie in G. Thus (3) implies (4).

Finally, suppose that (4) holds, in other words, $-\log \operatorname{dist}(z, bG)$ is plurisubharmonic. To get (1), all that needs to be done is to modify this function to make it smooth and *strictly* plurisubharmonic. Here are the technical details.

To start, let $u(z)$ denote $\max(|z|^2, -\log \operatorname{dist}(z, bG))$. Then u is a continuous, plurisubharmonic exhaustion function for G. By adding a constant to u, one may assume that the minimum value of u on G is 0. For each positive integer j, let G_j denote the subset of G on which $u < j$. The sets G_j form an increasing sequence of relatively compact open subsets of G.

Extend u to be 0 outside G, and convolve u with a smooth mollifying function with small support to get an infinitely differentiable function on $\mathbb{C}^n$ that is plurisubharmonic on a neighborhood of the closure of G_j and that closely approximates u from above on that neighborhood. Adding $\varepsilon_j |z|^2$ for a suitably small positive constant ε_j gives a smooth function u_j on $\mathbb{C}^n$, strictly plurisubharmonic on a neighborhood of the closure of G_j, such that $u < u_j < u + 1$ on G_j. It remains to splice the functions u_j together to get the required smooth, strictly plurisubharmonic exhaustion function for G.

A natural way to build the final function is to use an infinite series. A simple way to guarantee that the sum remains infinitely differentiable is to make the series locally finite. To carry out this plan, let χ be an infinitely differentiable, convex function of one real variable such that $\chi(t) = 0$ when $t \leq 0$, and both χ' and χ'' are positive when $t > 0$.

Exercise 17. Verify that an example of such a function χ is

$$\begin{cases} 0, & \text{if } t \leq 0, \\ e^t e^{-1/t}, & \text{if } t > 0. \end{cases}$$

Exercise 18. Show that if φ is any increasing convex function of one real variable, and if v is any plurisubharmonic function, then the composite function $\varphi \circ v$ is plurisubharmonic. Moreover, if φ is strictly convex and v is strictly plurisubharmonic, then $\varphi \circ v$ is strictly plurisubharmonic.

The remainder of the proof consists of inductively choosing positive constants c_j to make $\sum_{j=1}^{\infty} c_j \chi(u_j(z) - j + 1)$ have the required properties. The induction statement is that on the set G_k, the sum $\sum_{j=1}^{k} c_j\chi(u_j(z) - j + 1)$ is strictly plurisubharmonic and larger than $u(z)$.

For the basis step ($k = 1$), observe that u_1 is strictly larger than u on G_1 and hence is strictly positive there. By Exercise 18, the composite function $\chi \circ u_1$ is strictly plurisubharmonic on G_1. Take the constant c_1 so large that $c_1\chi \circ u_1$ exceeds u on G_1.

Suppose now that the induction statement holds for the integer k. If z is in G_{k+1} but outside G_k, then $k \le u(z) < u_{k+1}(z)$; for such z, the function $\chi(u_{k+1}(z) - k)$ is positive and strictly plurisubharmonic. By multiplying by a suitably large constant c_{k+1}, it is possible to arrange for $\sum_{j=1}^{k+1} c_j\chi(u_j(z)-j+1)$ to be both strictly plurisubharmonic and larger than $u(z)$ when z is in G_{k+1} but outside G_k. Since the function $\chi(u_{k+1}(z) - k)$ is non-negative and (weakly) plurisubharmonic on all of G_{k+1}, the induction hypothesis implies that the sum of $k+1$ terms is strictly plurisubharmonic and larger than u on all of G_{k+1}.

It remains to check that the infinite series does converge to an infinitely differentiable function on G. This property is local, so it is enough to check on a ball whose closure is contained in G and hence in some G_m. If $j \ge m+2$, then $\chi(u_j(z) - j + 1) = 0$ when $z \in G_m$ (since $u_j < u{+}1$ on G_j), so only finitely many terms of the sum contribute on the ball. Hence the series converges to an infinitely differentiable function; the preceding paragraph shows that the limit function is strictly plurisubharmonic; since the sum exceeds the exhaustion function u, it is an exhaustion function too. □

3.3 The Levi problem

The characterizations of pseudoconvexity considered so far are essentially internal to the domain. Eugenio Elia Levi discovered[14] a characterization of pseudoconvexity that involves the differential geometry of the boundary of the domain. This condition requires the boundary to be a twice continuously differentiable manifold. Since Levi's condition is local, one ought first to observe that pseudoconvexity is a local property of the boundary.

Theorem 9. *A domain G in $\mathbb{C}^n$ is pseudoconvex if and only if each boundary point of G has an open neighborhood U in $\mathbb{C}^n$ such that the intersection $U \cap G$ is pseudoconvex.*

Proof. If G is pseudoconvex, and B is a ball centered at a boundary point, then the intersection $B \cap G$ is pseudoconvex because $\max(-\log \operatorname{dist}(z, bB), -\log \operatorname{dist}(z, bG))$ is a plurisubharmonic exhaustion function.

Conversely, suppose U is a neighborhood of a boundary point p such that $U \cap G$ is pseudoconvex, that is, such that $-\log \operatorname{dist}(z, b(U \cap G))$ is plurisubharmonic on $U \cap G$. If z is closer to p than to bU, then $\operatorname{dist}(z, bG) = \operatorname{dist}(z, b(U \cap G))$. Consequently, there is an open neighborhood V of bG such that $-\log \operatorname{dist}(z, bG)$ is plurisubharmonic for z in $V \cap G$. What remains to do is to modify this function to get a plurisubharmonic function in all of G. If G is bounded, then $G \setminus V$ is a compact set, and $-\log \operatorname{dist}(z, bG)$

[14] E. E. Levi, Studii sui punti singolari essenziali delle funzioni analitiche di due o più variabili complesse, *Annali di Matematica Pura ed Applicata (3)* **17** (1910) 61–87.

has an upper bound M on $G \setminus V$. Then $\max(M, -\log \operatorname{dist}(z, bG))$ is a plurisubharmonic exhaustion function for G.

If G is unbounded, then the set $G \setminus V$ is closed but not necessarily compact. For each non-negative real number r, the continuous function $-\log \operatorname{dist}(z, bG)$ has a maximum value on the intersection of $G \setminus V$ with the closed ball centered at 0 with radius r. By Exercise 19 below, there is a function $\varphi(|z|)$ that is plurisubharmonic on $\mathbb{C}^n$, that exceeds $-\log \operatorname{dist}(z, bG)$ when z is in $G \setminus V$, and that blows up at infinity. Then $\max(\varphi(|z|), -\log \operatorname{dist}(z, bG))$ is a plurisubharmonic exhaustion function for G, so G is pseudoconvex. □

Thus pseudoconvexity is a local property of the boundary of a domain. None of the properties so far shown to be equivalent to holomorphic convexity appears to be local. In particular, it is not evident how to get a globally defined holomorphic function that is singular at a boundary point given a locally defined holomorphic function that is singular at that point. The essence of the Levi problem—the equivalence between holomorphic convexity and pseudoconvexity—is to show that being a domain of holomorphy actually is a local property of the boundary of the domain.

Exercise 19. If g is a continuous function on $[0, \infty)$, then there is an increasing convex function φ such that $\varphi(t) > g(t)$ for all t.

3.3.1 The Levi form

Suppose that in a neighborhood of a boundary point of a domain there is a real-valued function ρ such that the boundary of the domain is the set where $\rho = 0$, the interior of the domain is the set where $\rho < 0$, and the exterior of the domain is the set where $\rho > 0$. Suppose additionally that ρ has continuous partial derivatives of second order and that the gradient of ρ is nowhere equal to 0 on the boundary of the domain. The implicit function theorem then implies that the boundary of the domain (in the specified neighborhood) is a twice differentiable (real) manifold. The abbreviation for this set of conditions is that the domain has "class C^2 boundary" or "class C^2 smooth boundary".

Levi's condition is that when z is in the boundary of the domain,

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \geq 0 \qquad \text{for vectors } w \text{ in } \mathbb{C}^n \text{ such that} \qquad \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} w_j = 0.$$

Notice that Levi's condition does not say that the function ρ is plurisubharmonic, because the inequality holds not for all vectors w in $\mathbb{C}^n$ but only for *complex tangent vectors* (vectors satisfying the side condition). The indicated Hermitian quadratic form, acting on the complex tangent space, is known as the *Levi form*. If the Levi form is strictly positive definite, then the domain is called *strictly pseudoconvex* (or strongly pseudoconvex).

Exercise 20. Although the Levi form depends on the choice of the defining function ρ, positivity (or non-negativity) of the Levi form is independent of the choice of defining function. Moreover, positivity (or non-negativity) of the Levi form is invariant under local biholomorphic changes of coordinates.

One can rephrase Levi's condition as the statement that there exists a positive constant C such that when z is in the boundary of the domain,

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}w_j\overline{w}_k + C\|w\|\left|\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}w_j\right| \ge 0 \qquad \text{for all vectors } w \text{ in } \mathbb{C}^n.$$

An advantage of this reformulation is that it eliminates the side condition about the complex tangent space: the inequality holds for *all* vectors w. The second formulation evidently implies the first statement of Levi's condition. To see, conversely, that Levi's condition implies the reformulation, decompose an arbitrary vector w into an orthogonal sum $w'+w''$, where $\sum_{j=1}^{n}\rho_j w'_j = 0$ (here ρ_j is a typographically convenient abbreviation for $\partial\rho/\partial z_j$), and $\sum_{j=1}^{n}\rho_j w''_j = \sum_{j=1}^{n}\rho_j w_j$. By hypothesis, the length of the gradient of ρ is bounded away from 0, so the length of the vector w'' is comparable to $\sum_{j=1}^{n}\rho_j w_j$. Substituting $w'+w''$ for w in Levi's condition shows that

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}w_j\overline{w}_k = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}w'_j\overline{w}'_k + O(\|w\|\,\|w''\|) \ge -C\|w\|\left|\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}w_j\right|$$

for some constant C, which is the reformulated version of the Levi condition.

Theorem 10. *A domain with class C^2 smooth boundary is pseudoconvex if and only if the Levi form is non-negative definite at each boundary point.*

Proof. First suppose that the domain G is pseudoconvex in the sense that the negative of the logarithm of the distance to the boundary of G is plurisubharmonic. A convenient function ρ to use as defining function is the signed distance to the boundary:

$$\rho(z) = \begin{cases} -\operatorname{dist}(z, bG), & z \in G, \\ +\operatorname{dist}(z, bG), & z \notin G. \end{cases}$$

It follows from the implicit function theorem that this defining function is class C^2 close to the boundary of G. By hypothesis, the complex Hessian of $-\log|\rho|$ is non-negative inside G:

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\left(-\frac{1}{\rho}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k} + \frac{1}{\rho^2}\frac{\partial\rho}{\partial z_j}\frac{\partial\rho}{\partial\overline{z}_k}\right)w_j\overline{w}_k \ge 0 \qquad \text{for every } w \text{ in } \mathbb{C}^n.$$

Since $-1/\rho$ is positive at points inside the domain, it follows that

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}w_j\overline{w}_k \ge 0 \qquad \text{when } z \text{ is inside and} \qquad \sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}w_j = 0.$$

As observed above, this Levi condition is equivalent to the existence of a positive constant C such that

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_k}w_j\overline{w}_k + C\|w\|\left|\sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}w_j\right| \ge 0 \qquad \text{for all vectors } w \text{ in } \mathbb{C}^n.$$

The constant C depends on the maximum of the second derivatives of ρ and the maximum of $1/|\nabla\rho|$, and these quantities are bounded near bG by hypothesis. Since ρ has continuous second derivatives, it is possible to take the limit as z approaches the boundary, which gives Levi's condition on the boundary.

For the converse, it is necessary to deduce the existence of a plurisubharmonic exhaustion function from Levi's condition. In view of Theorem 9, it will suffice to work locally on, say, a small ball that intersects the boundary of the domain.

The implicit function theorem implies that a class C^2 boundary can be written locally as the graph of a twice continuously differentiable real-valued function. After a complex linear change of coordinates, a local defining function ρ takes the form $\varphi(\operatorname{Re} z_1, \operatorname{Im} z_1, \ldots, \operatorname{Re} z_{n-1}, \operatorname{Im} z_{n-1}, \operatorname{Re} z_n) - \operatorname{Im} z_n$. The hypothesis says that there exists a positive constant C such that at boundary points,

$$\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k + 2C\|w\| \left| \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} w_j \right| \geq 0 \qquad \text{for all vectors } w \text{ in } \mathbb{C}^n$$

(the factor of 2 being inserted for later convenience). Since ρ depends linearly on $\operatorname{Im} z_n$, derivatives of ρ are independent of $\operatorname{Im} z_n$. Thus the preceding condition holds not only on the boundary of G but also off the boundary, in a ball in $\mathbb{C}^n$ (perhaps a smaller ball than the initial one).

Let u denote $-\log|\rho|$; the goal is to modify the function u to get a local plurisubharmonic function in G that blows up at the boundary. At points inside G, the same calculation as above shows for every vector w in $\mathbb{C}^n$ that

$$\begin{aligned}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k &= \frac{1}{|\rho|} \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k + \frac{1}{\rho^2} \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial \rho}{\partial z_j} \frac{\partial \rho}{\partial \overline{z}_k} w_j \overline{w}_k \\ &\geq -\frac{2C}{|\rho|}\,\|w\| \left| \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} w_j \right| + \frac{1}{\rho^2} \left| \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} w_j \right|^2.\end{aligned}$$

Since $-2ab \geq -a^2 - b^2$, it follows that

$$\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \geq -C^2\,\|w\|^2.$$

The preceding inequality implies that $u(z) + C^2\|z\|^2$ is a plurisubharmonic function in the intersection of G with a small ball B, and this function blows up at the boundary of G. If the ball B has center a and radius r, then $\max(-\log(r - \|z-a\|), u(z) + C^2\|z\|^2)$ is a plurisubharmonic exhaustion function for $B \cap G$. Thus G is locally pseudoconvex near every boundary point, so by Theorem 9, the domain G is pseudoconvex. □

Knowing Levi's condition makes it possible to rephrase the notion of pseudoconvexity in the following way.

Theorem 11. *A domain is pseudoconvex if and only if it can be expressed as the union of an increasing sequence of class C^∞ smooth domains each of which is locally biholomorphically equivalent to a strongly convex domain.*

Proof. A convex domain is pseudoconvex, and pseudoconvexity is a local property that is biholomorphically invariant, so a domain that is locally equivalent to a convex domain is pseudoconvex. The increasing union of pseudoconvex domains is pseudoconvex by, for instance, version (b) of the Kontinuitätssatz. Thus one direction of the theorem follows by putting together prior results.

Conversely, suppose that G is pseudoconvex. Then G admits a C^∞, strictly plurisubharmonic exhaustion function u. Fix a base point in G and consider the connected component containing that point of the sub-level set where $u < c$. By Sard's theorem,[15] the gradient of u is non-zero on the set where $u = c$ for most values of c (all but a set of values of c of measure zero in $\mathbb{R}$). Thus G is exhausted by an increasing sequence of C^∞ strictly pseudoconvex domains.

It remains to show that each smooth level set where the strictly plurisubharmonic function u equals c is locally equivalent to a strongly convex domain via a local biholomorphic mapping. Fix a point a such that $u(a) = c$, and consider the Taylor expansion of $u(z) - c$ in a neighborhood of a:

$$\begin{aligned} 2\operatorname{Re}\Bigg[\sum_{j=1}^{n} \frac{\partial u}{\partial z_j}(a)(z_j - a_j) + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial z_k}(a)(z_j - a_j)(z_k - a_k)\Bigg] \\ + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(a)(z_j - a_j)(\overline{z}_k - \overline{a}_k) + O(\|z - a\|^3). \end{aligned} \tag{3.1}$$

The expression whose real part appears on the first line is a holomorphic function of z whose gradient is nonzero at the point a, so this function will serve as the first coordinate w_1 of a local biholomorphic change of coordinates $w(z)$ such that $w(a) = 0$. In a neighborhood of the point a, the level surface where $u(z) - c = 0$ has a defining function $\rho(w)$ in the new coordinates of the form

$$2\operatorname{Re} w_1 + \sum_{j=1}^{n}\sum_{k=1}^{n} L_{jk} w_j \overline{w}_k + O(\|w\|^3),$$

where the matrix L_{jk} is a positive definite Hermitian matrix corresponding to the positive definite matrix $u_{j\overline{k}}$ in the new coordinates. Thus the quadratic part of the real Taylor expansion of ρ in the real coordinates corresponding to w is positive definite, which means that the level set where $\rho = 0$ is strongly convex in the real sense. □

Exercise 21. Solve the Levi problem for complete Reinhardt domains in $\mathbb{C}^2$ by showing that Levi's condition in that setting is equivalent to logarithmic convexity.

[15] Arthur Sard, The measure of the critical values of differentiable maps, *Bulletin of the American Mathematical Society* **48** (1942) 883–890. Since the function u takes values in $\mathbb{R}^1$, the claim already follows from an earlier result of Anthony P. Morse, The behavior of a function on its critical set, *Annals of Mathematics (2)* **40** (1939), no. 1, 62–70.

3.3.2 Applications of the $\overline{\partial}$ problem

Theorem 9 shows that pseudoconvexity is a local property of the boundary of a domain, but it is far from obvious that holomorphic convexity is a local property of the boundary. To solve the Levi problem for a general pseudoconvex domain, one needs some technical machinery to forge the connection between the local and the global. One approach is sheaf theory, another is integral representations, and a third is the $\overline{\partial}$-equation. The following discussion uses the third method, for the idea seems the most intuitive.

Some notation is needed: if f is a function, then $\overline{\partial} f$ denotes $\sum_{j=1}^{n}(\partial f/\partial \overline{z}_j)\, d\overline{z}_j$, a so-called $(0,1)$-form. A function f is holomorphic precisely when $\overline{\partial} f = 0$. The question of interest here is whether a given $(0,1)$-form α, say $\sum_{j=1}^{n} a_j(z)\, d\overline{z}_j$, can be written as $\overline{\partial} f$ for some function f. In order for the mixed second partial derivatives of f to be equal, it must happen that $\partial a_j/\partial \overline{z}_k = \partial a_k/\partial \overline{z}_j$ for all j and k; these necessary conditions are abbreviated as $\overline{\partial}\alpha = 0$; in words, the form α is $\overline{\partial}$-closed.

The key ingredient in solving the Levi problem is the following solvability theorem for the inhomogeneous Cauchy-Riemann equations.

Theorem 12. *Let G be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^∞ smooth boundary. If α is a $\overline{\partial}$-closed $(0,1)$-form with C^∞ coefficients in G, then there exists a C^∞ function f in G such that $\overline{\partial} f = \alpha$.*

The conclusion holds without any hypothesis about boundary smoothness, but then the proof is more technical. For present purposes, it suffices even to prove the theorem under the additional hypothesis of strong pseudoconvexity.

Solution of the Levi problem for bounded strongly pseudoconvex domains

Granted Theorem 12, one can easily solve the Levi problem for the approximating strongly pseudoconvex domains arising in the proof of Theorem 11. Indeed, let G be a bounded domain with boundary defined by an infinitely differentiable strictly plurisubharmonic function; it is not necessary to assume here that the gradient of the defining function is non-zero on the boundary. To show that G is a domain of holomorphy, it suffices to produce a global holomorphic function on G that is singular at a specified boundary point p. The proof of Theorem 11 provides a local holomorphic function f_p defined in a neighborhood of p that is equal to 0 at p and that has no zero on $\overline{G}\setminus\{p\}$. Indeed, the holomorphic function whose real part appears in the first line of formula (3.1) will serve for f_p, even on those level sets of the strictly plurisubharmonic function u that happen to be non-smooth. The goal is to modify the reciprocal $1/f_p$ to produce a globally defined function on G that is singular at p.

Let χ be a smooth, real-valued, non-negative cut-off function that is identically equal to 1 in a neighborhood of p and identically equal to 0 outside a larger neighborhood (contained in the set where f_p is defined). The function χ/f_p is defined globally on G and blows up at p, but χ/f_p is not globally holomorphic. The theorem on solvability of the $\overline{\partial}$-equation makes it possible to adjust this function to get a holomorphic function.

Indeed, the $(0,1)$ form $(\overline{\partial}\chi)/f_p$ is identically equal to 0 in a neighborhood of p, and since the zero set of f_p touches $\overline{G}$ only at p, this form has C^∞ coefficients in a neighborhood D of $\overline{G}$, which may be taken to be a strongly pseudoconvex domain with C^∞ smooth boundary. The form $(\overline{\partial}\chi)/f_p$ is $\overline{\partial}$-closed on this domain D, since $\overline{\partial}\chi$ is $\overline{\partial}$-closed, and $1/f_p$ is holomorphic away from the zeroes of f_p. Theorem 12 produces a C^∞ function v on D such that $\overline{\partial}v = (\overline{\partial}\chi)/f_p$.

Consequently, the function $v - (\chi/f_p)$ is a holomorphic function on the set where $f_p \neq 0$, in particular, on $\overline{G} \setminus \{p\}$. The function v, being smooth on $\overline{G}$, is bounded there, so the holomorphic function $v - (\chi/f_p)$ is singular at p. Thus there exists a holomorphic function on all of G that is singular at a prescribed boundary point, so G is a domain of holomorphy.

This argument solves the Levi problem for bounded strongly pseudoconvex domains, modulo the proof of solvability of the $\overline{\partial}$-equation.

It is worthwhile noticing that the preceding argument essentially contains the existence of a *peak function* on the strongly pseudoconvex domain G, that is, a function h holomorphic on $\overline{G}$ such that $h(p) = 1$, while $|h(z)| < 1$ when $z \in \overline{G} \setminus \{p\}$. In fact, one may as well assume that $\operatorname{Re} f_p < 0$ on the part of $\overline{G} \setminus \{p\}$ inside the neighborhood where f_p is defined (see formula (3.1)). Let g denote $1/[c + v - (\chi/f_p)]$, where c is a real constant larger than the maximum of $|v|$ on $\overline{G}$. Then g is well defined and holomorphic on $\overline{G} \setminus \{p\}$, because the denominator has positive real part (and so is not 0). On the other hand, in a neighborhood of p, the function g equals $f_p/[(c + v)f_p - 1]$, so g is holomorphic also in a neighborhood of p and equals 0 at p. Thus g is holomorphic in a neighborhood of $\overline{G}$, it has positive real part on $\overline{G} \setminus \{p\}$, and it equals 0 at p. Therefore e^{-g} serves as the required holomorphic peak function h.

Proof of the Oka-Weil theorem

Another application of the solvability of the $\overline{\partial}$-equation on strongly pseudoconvex domains is the Oka-Weil theorem (Theorem 4). Indeed, the tools are at hand to prove the following generalization.

Theorem 13. *If G is a domain of holomorphy in $\mathbb{C}^n$, and K is a compact subset of G that is convex with respect to the holomorphic functions on G, then every function holomorphic in a neighborhood of K can be approximated uniformly on K by functions holomorphic on G.*

Theorem 4 follows by taking G equal to $\mathbb{C}^n$, because convexity with respect to entire functions is the same as polynomial convexity, and approximation by entire functions is equivalent to approximation by polynomials.

Proof of Theorem 13. Suppose f is holomorphic in an open neighborhood U of K, and ε is a specified positive number. The goal is to approximate f on K within ε by functions that are holomorphic on the domain G.

Let L be a compact subset of G containing U and convex with respect to $\mathcal{O}(G)$. The initial goal is to show that f can be approximated on K within ε by functions that are holomorphic in a neighborhood of L; a subsequent limiting argument as L expands will finish the proof.

Fix an open neighborhood V of L having compact closure in G. The first observation is that there are finitely many holomorphic functions $f_1, \ldots, f_k$ on G such that

$$K \subseteq \{\, z \in V : |f_1(z)| \leq 1, \ldots, |f_k(z)| \leq 1 \,\} \subset U.$$

In other words, the compact set K can be closely approximated from outside by a compact analytic polyhedron defined by functions that are holomorphic on G. The reason is similar to the proof of Theorem 3: the set $\overline{V} \setminus U$ is compact, and each point of this set can be separated from the holomorphically convex set K by a function holomorphic on G, so a compactness argument furnishes a finite number of separating functions. Hence one might as well assume from the start that K is equal to the indicated analytic polyhedron.

For the same reason, the holomorphically convex compact set L can be approximated from outside by a compact analytic polyhedron contained in V and defined by a finite number of functions holomorphic on G. Again, one might as well assume that L equals that analytic polyhedron.

The main step in the proof is to show that functions holomorphic in a neighborhood of L are dense in the functions holomorphic in a neighborhood of $\{\, z \in L : |f_1(z)| \leq 1 \,\}$. An evident induction on the number of functions defining the polyhedron K then implies that $\mathcal{O}(L)$ is dense in $\mathcal{O}(K)$.

It is here that Oka's great insight enters: he had the idea that raising the dimension by looking at the graph of f_1 can simplify matters. Let L_1 denote $\{\, z \in L : |f_1(z)| \leq 1 \,\}$, and let D denote the closed unit disc in $\mathbb{C}$. The claim is that if g is a holomorphic function in a neighborhood of L_1, then there is a corresponding function $F(z, w)$ in $\mathbb{C}^{n+1}$, holomorphic in a neighborhood of $L \times D$, such that $g(z) = F(z, f_1(z))$ when z is in a neighborhood of L_1. In other words, there is a holomorphic function on all of $L \times D$ whose restriction to the graph of f_1 recovers g on L_1.

How does this construction help? The point is that F can be expanded in the last variable in a Maclaurin series, $F(z, w) = \sum_{j=0}^{\infty} a_j(z) w^j$, in which the coefficient functions a_j are holomorphic on L. Then $g(z) = \sum_{j=0}^{\infty} a_j(z) f_1(z)^j$ in a neighborhood of L_1, and the partial sums of this series are holomorphic functions on L that uniformly approximate g on L_1.

To construct F, take an infinitely differentiable cut-off function χ in $\mathbb{C}^n$ that is identically equal to 1 in a neighborhood of L_1 and that is identically equal to 0 outside a slightly larger neighborhood (contained in the set where g is defined). Consider in $\mathbb{C}^{n+1}$ the $(0, 1)$-form

$$\frac{g\,\overline{\partial}\chi}{f_1(z) - w}, \qquad \text{where } z \in \mathbb{C}^n, \text{ and } w \in \mathbb{C}.$$

The only points z where $\overline{\partial}\chi(z) \neq 0$ are outside L_1, where $|f_1(z)| > 1$, so this $(0, 1)$-form is well defined and smooth on a neighborhood of $L \times D$. Evidently the form is $\overline{\partial}$-closed. The compact analytic polyhedron L can be approximated from outside by open analytic polyhedra, so $L \times D$ can be approximated from outside by domains of holomorphy (more precisely, each connected component can be so approximated). By Theorem 11, the compact set $L \times D$ can be approximated from outside by bounded, smooth, strongly pseudoconvex open sets. Consequently, the solution of the $\overline{\partial}$-equation supplies a smooth function v in a neighborhood of $L \times D$ such that $g(z)\chi(z) - v(z,w)(f_1(z) - w)$ is holomorphic on $L \times D$. This holomorphic function is the required function $F(z,w)$ on $L \times D$ such that $F(z, f_1(z)) = g(z)$ on L_1.

The proof is now complete that $\mathcal{O}(L)$ is dense in $\mathcal{O}(K)$. It remains to approximate a function holomorphic in a neighborhood of K by a function holomorphic on all of G. To this end, let $\{K_j\}_{j=0}^{\infty}$ be an exhaustion of G by an increasing sequence of holomorphically convex compact subsets of G, each containing an open neighborhood of the preceding one, where K_0 may be taken equal to K. By what has already been proved, $\mathcal{O}(K_{j+1})$ is dense in $\mathcal{O}(K_j)$ for every positive integer j. Suppose given a function f holomorphic in a neighborhood of K_0 and a positive ε. There is a function h_1 holomorphic in a neighborhood of K_1 such that $|f - h_1| < \varepsilon/2$ on K_0, and one can choose inductively a sequence of functions h_j such that h_j is holomorphic on K_j, and $|h_j - h_{j+1}| < \varepsilon/2^{j+1}$ on K_j. The telescoping series $h_1 + \sum_{j=1}^{\infty}(h_{j+1} - h_j)$ then converges uniformly on every compact subset of G to a holomorphic function that approximates f within ε on K_0. $\square$

Solution of the Levi problem for arbitrary pseudoconvex domains

What has been shown so far is that if G is a pseudoconvex domain, then there exists an infinitely differentiable strictly plurisubharmonic exhaustion function u, and the Levi problem is solvable for the sublevel sets of u, which are thus domains of holomorphy. A limiting argument is needed to show that G itself is a domain of holomorphy.

For each real number r, let G_r denote the sublevel set $\{\, z \in G : u(z) < r \,\}$, and let $\overline{G}_r$ denote its closure, the set $\{\, z \in G : u(z) \le r \,\}$. The key lemma is that $\mathcal{O}(G_t)$ is dense in $\mathcal{O}(\overline{G}_r)$ when $t > r$.

To confirm this density, it suffices by Theorem 13 to show that the compact set $\overline{G}_r$ is convex with respect to the holomorphic functions on G_t. If it were not, then its holomorphic hull (which is a compact subset of G_t because G_t is a domain of holomorphy) would contain a point p where the restriction of the exhaustion function u to the hull assumes a maximal value s such that $r < s < t$. As observed on page 39, there is a holomorphic peak function h for G_s at p such that $h(p) = 1$ while $|h(z)| < 1$ when $z \in G_s$. Since h is holomorphic in a neighborhood of the holomorphic hull of $\overline{G}_r$ (and this hull is by definition a holomorphically convex subset of G_t), Theorem 13 implies that h can be approximated on this hull by functions holomorphic on G_t. Since h separates p from $\overline{G}_r$, so do holomorphic functions on G_t, and therefore p is not in the holomorphic hull of $\overline{G}_r$ after all. Thus $\overline{G}_r$ is $\mathcal{O}(G_t)$-convex, and $\mathcal{O}(G_t)$ is dense in $\mathcal{O}(\overline{G}_r)$.

The same argument with a telescoping series as in the final paragraph of the proof of Theorem 13 now shows that $\mathcal{O}(G)$ is dense in $\mathcal{O}(G_r)$ for every r.

To see that G is a domain of holomorphy, fix a compact subset K. What needs to be shown is that $\widehat{K}_G$ is a compact subset of G. Fix a sufficiently large real number r such that K is a compact subset of G_r. Since G_r is a domain of holomorphy, the hull $\widehat{K}_{G_r}$ is a compact subset of G_r. It suffices to show that $\widehat{K}_G = \widehat{K}_{G_r}$. It is automatic that

$\widehat{K}_{G_r} \subseteq \widehat{K}_G$, so what needs to be shown is that $\widehat{K}_G \subseteq \widehat{K}_{G_r}$. In other words, if $p \notin \widehat{K}_{G_r}$, one needs to find a holomorphic function on G that separates p from K.

If $p \in G_r$, then there is no difficulty, because there is a holomorphic function on G_r that separates p from K, and $\mathcal{O}(G)$ is dense in $\mathcal{O}(G_r)$. If p is not in G_r, then p is in G_s for some value of s larger than r. Since $\mathcal{O}(G_s)$ is dense in $\mathcal{O}(G_r)$, the intersection of $\widehat{K}_{G_s}$ with G_r equals $\widehat{K}_{G_r}$. Therefore the function that is identically equal to 0 in a neighborhood of $\widehat{K}_{G_r}$ and identically equal to 1 outside G_r is holomorphic on $\widehat{K}_{G_s}$. By Theorem 13, this function can be approximated on $\widehat{K}_{G_s}$ by functions holomorphic on G_s and hence (since $\mathcal{O}(G)$ is dense in $\mathcal{O}(G_s)$) by functions holomorphic on G. Thus the point p can be separated from K by functions holomorphic on G, so p is not in $\widehat{K}_G$.

The argument has shown that the pseudoconvex domain G is holomorphically convex, so G is a domain of holomorphy. The solution of the Levi problem for pseudoconvex domains is now complete, except for proving the solvability of the $\overline{\partial}$-equation on bounded strongly pseudoconvex domains with smooth boundary.

3.3.3 Solution of the $\overline{\partial}$-equation on smooth pseudoconvex domains

The resolution of the Levi problem required knowing that the $\overline{\partial}$-equation on a bounded, (strongly) pseudoconvex domain with smooth boundary is solvable. The following discussion will prove this solvability using ideas developed in the 1950s and 1960s by Charles B. Morrey, Donald C. Spencer, Joseph J. Kohn, and Lars Hörmander.

The method is based on Hilbert space techniques. The relevant Hilbert space is $L^2(G)$, the space of square-integrable functions on G with inner product $\langle f, g\rangle$ equal to $\int_G f\,\overline{g}\,dV$, where dV denotes Lebesgue volume measure. The inner product extends to forms by summing the inner products of components of the forms.

The operator $\overline{\partial}$ acts on square-integrable functions in the sense of distributions, so one can view $\overline{\partial}$ as an unbounded operator from $L^2(G)$ to the space of $(0,1)$-forms with $L^2(G)$ coefficients. The domain of $\overline{\partial}$ is the subspace of functions f for which the distributional coefficients of $\overline{\partial} f$ are represented by square-integrable functions. Since compactly supported, infinitely differentiable functions are dense in $L^2(G)$, the operator $\overline{\partial}$ is a densely defined operator, and it is easy to see that this operator is a closed operator. Consequently, there is a Hilbert space adjoint $\overline{\partial}^*$, which too is a closed, densely defined operator.

If f is a $(0,1)$-form $\sum_{j=1}^n f_j\, d\overline{z}_j$, then

$$\overline{\partial} f = \sum_{1\le j<k\le n} \left(\frac{\partial f_j}{\partial \overline{z}_k} - \frac{\partial f_k}{\partial \overline{z}_j} \right) d\overline{z}_k \wedge d\overline{z}_j.$$

If you are unfamiliar with differential forms, then you can view the preceding expression as simply a formal gadget that is a convenient notation for stating the necessary condition for solvability of the equation $\overline{\partial} u = f$: namely, that $\overline{\partial} f = 0$. The goal is to show that this necessary condition is sufficient on bounded pseudoconvex domains with C^∞ smooth boundary; moreover, the solution u should be infinitely differentiable if the coefficients of f are infinitely differentiable. (The solution u is not unique, because any holomorphic function can be subtracted from u, but if one solution is an infinitely differentiable function in G, then every solution is infinitely differentiable.)

Reduction to an estimate

The claim is that the whole problem boils down to proving the following *basic estimate*: there exists a constant C such that for every $(0,1)$-form that belongs to both the domain of $\overline{\partial}$ and the domain of $\overline{\partial}^*$, one has that

$$\|f\|^2 \le C(\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2). \tag{3.2}$$

It will turn out that the constant C depends on the diameter of the domain, so boundedness of the domain is important. Why does this estimate imply solvability of the $\overline{\partial}$-equation?

Suppose that g is a specified $\overline{\partial}$-closed $(0,1)$-form with coefficients in $L^2(G)$. Consider the mapping $\overline{\partial}^* f \mapsto \langle f, g\rangle$ for $(0,1)$-forms f that belong to both the domain of $\overline{\partial}^*$ and the kernel of $\overline{\partial}$. The basic estimate implies that $\|\overline{\partial}^* f\|$ dominates $\|f\|$, so this mapping is a well defined bounded linear operator on the subspace $\overline{\partial}^*(\operatorname{dom} \overline{\partial}^* \cap \ker \overline{\partial})$ of $L^2(G)$.

The Riesz representation theorem produces a function u such that $\langle \overline{\partial}^* f, u\rangle = \langle f, g\rangle$ for all f in the intersection of the domain of $\overline{\partial}^*$ and the kernel of $\overline{\partial}$. On the other hand, if f is in the the intersection of the domain of $\overline{\partial}^*$ and the orthogonal complement of the kernel of $\overline{\partial}$, then the same equality holds trivially because sides are zero (namely, $\langle f, g\rangle = 0$ because g is in the kernel of $\overline{\partial}$; similarly $\langle f, \overline{\partial}\varphi\rangle = 0$ for every infinitely differentiable, compactly supported function φ, so $\langle \overline{\partial}^* f, \varphi\rangle = 0$, and therefore $\overline{\partial}^* f = 0$). Consequently, u is in the domain of the adjoint of $\overline{\partial}^*$, hence in the domain of $\overline{\partial}$, and $\langle f, \overline{\partial} u\rangle = \langle f, g\rangle$ for all f in the domain of $\overline{\partial}^*$. Since the operator $\overline{\partial}^*$ is densely defined, it follows that $\overline{\partial} u = g$.

Thus the basic estimate implies the existence of a solution of the $\overline{\partial}$-equation in $L^2(G)$. Additionally, the solution u is supposed to be infinitely differentiable in G when g has coefficients that are infinitely differentiable functions in G. By Sobolev's lemma (or the Sobolev embedding theorem) from functional analysis, it suffices to show that all distributional derivatives of u are locally square integrable. By hypothesis, each $\partial u/\partial \overline{z}_j$ and all its derivatives are locally square integrable, so what needs to be shown is that $\partial^{|\beta|} u/\partial z^\beta$ is locally square integrable for each multi-index β. Equivalently, it is enough to show that for every infinitely differentiable, real-valued function φ having compact support in G, the integral

$$\int_G \varphi \frac{\partial^{|\beta|} u}{\partial z^\beta} \overline{\frac{\partial^{|\beta|} u}{\partial z^\beta}}\, dV$$

is finite. Integrating all the derivatives by parts results in a sum of integrals involving only barred derivatives of u, and these derivatives are already under control. Hence all derivatives of u are square-integrable on compact subsets of G. Thus the solution u is infinitely differentiable in G when g is. (The catchphrase here is "interior elliptic regularity".)

The much more difficult question of whether the derivatives of the solution u extend to the boundary of the domain G when g has this property is beyond the scope of these notes. This question of boundary regularity is the subject of current research, and the situation is not completely understood.

Proof of the basic estimate

The cognoscenti sometimes describe the proof of the basic estimate as "an exercise in integration by parts". This characterization becomes less of an exaggeration if you admit Stokes's theorem as an instance of integration by parts.

The plan is to work on the right-hand side of the basic estimate. It is convenient to prove the estimate for $(0,1)$-forms whose coefficients are sufficiently smooth functions on the closure of G. There is a technical point that needs attention here: namely, to prove that such forms are dense in the intersection of the domains of $\overline{\partial}$ and $\overline{\partial}^*$. That the necessary density does hold is a special case of the so-called Friedrichs lemma, a general construction of Kurt Friedrichs.[16]

Suppose, then, that $f = \sum_{j=1}^n f_j \, d\overline{z}_j$, and the f_j are smooth functions on the closure of G. Since

$$\begin{aligned} |\overline{\partial} f|^2 &= \sum_{1\le j<k\le n} \left| \frac{\partial f_j}{\partial \overline{z}_k} - \frac{\partial f_k}{\partial \overline{z}_j} \right|^2 = \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n \left| \frac{\partial f_j}{\partial \overline{z}_k} - \frac{\partial f_k}{\partial \overline{z}_j} \right|^2 \\ &= \sum_{j=1}^n \sum_{k=1}^n \left(\left| \frac{\partial f_j}{\partial \overline{z}_k} \right|^2 - \frac{\partial f_j}{\partial \overline{z}_k} \overline{\frac{\partial f_k}{\partial \overline{z}_j}} \right), \end{aligned}$$

it follows that

$$\|\overline{\partial} f\|^2 = \sum_{j=1}^n \sum_{k=1}^n \int_G \left(\left| \frac{\partial f_j}{\partial \overline{z}_k} \right|^2 - \frac{\partial f_j}{\partial \overline{z}_k} \overline{\frac{\partial f_k}{\partial \overline{z}_j}} \right) dV. \tag{3.3}$$

To analyze $\|\overline{\partial}^* f\|$ requires a formula for $\overline{\partial}^* f$. If u is a smooth function on the closure of G, and ρ is a defining function for G normalized such that $|\nabla \rho| = 1$ on bG, then

$$\begin{aligned} \langle \overline{\partial}^* f, u \rangle = \langle f, \overline{\partial} u \rangle &= \int_G \sum_{j=1}^n f_j \overline{\frac{\partial u}{\partial \overline{z}_j}} \, dV \\ &= \int_G \sum_{j=1}^n -\frac{\partial f_j}{\partial z_j} \, \overline{u} \, dV + \int_{bG} \sum_{j=1}^n f_j \frac{\partial \rho}{\partial z_j} \, \overline{u} \, dS, \end{aligned}$$

where dS denotes $(2n-1)$-dimensional Lebesgue measure on the boundary of G. Since u is arbitrary, it follows that f is in the domain of $\overline{\partial}^*$ if and only if

$$\sum_{j=1}^n f_j \frac{\partial \rho}{\partial z_j} = 0 \qquad \text{on the boundary of } G, \tag{3.4}$$

and then $\overline{\partial}^* f = -\sum_{j=1}^n \partial f_j / \partial z_j$.

[16] K. O. Friedrichs, The identity of weak and strong extensions of differential operators, *Transactions of the American Mathematical Society* **55**, no. 1, (1944) 132–151.

Now integrate by parts:

$$\|\overline{\partial}^* f\|^2 = \sum_{j=1}^n \sum_{k=1}^n \int_G \frac{\partial f_j}{\partial z_j} \overline{\frac{\partial f_k}{\partial z_k}}\, dV = -\sum_{j=1}^n \sum_{k=1}^n \int_G \frac{\partial^2 f_j}{\partial z_j \partial \overline{z}_k}\, \overline{f}_k \, dV,$$

where the boundary term vanishes because f satisfies the boundary condition (3.4) for membership in the domain of $\overline{\partial}^*$. Integrating by parts a second time shows that

$$\|\overline{\partial}^* f\|^2 = \sum_{j=1}^n \sum_{k=1}^n \int_G \frac{\partial f_j}{\partial \overline{z}_k} \overline{\frac{\partial f_k}{\partial \overline{z}_j}}\, dV - \sum_{j=1}^n \sum_{k=1}^n \int_{bG} \frac{\partial f_j}{\partial \overline{z}_k}\, \overline{f}_k \, \frac{\partial \rho}{\partial z_j}\, dS.$$

The boundary condition (3.4) implies that the differential operator $\sum_{k=1}^n (\overline{f}_k)(\partial/\partial \overline{z}_k)$ is a tangential differential operator, so applying this operator to (3.4) shows that on the boundary,

$$0 = \sum_{k=1}^n \overline{f}_k \frac{\partial}{\partial \overline{z}_k} \Bigg(\sum_{j=1}^n f_j \frac{\partial \rho}{\partial z_j} \Bigg) = \sum_{j=1}^n \sum_{k=1}^n \Bigg(\overline{f}_k \frac{\partial f_j}{\partial \overline{z}_k} \frac{\partial \rho}{\partial z_j} + f_j \overline{f}_k \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} \Bigg).$$

Combining this identity with the preceding equation shows that

$$\begin{aligned} \|\overline{\partial}^* f\|^2 &= \sum_{j=1}^n \sum_{k=1}^n \int_G \frac{\partial f_j}{\partial \overline{z}_k} \overline{\frac{\partial f_k}{\partial \overline{z}_j}}\, dV + \sum_{j=1}^n \sum_{k=1}^n \int_{bG} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} f_j \overline{f}_k \, dS \\ &\geq \sum_{j=1}^n \sum_{k=1}^n \int_G \frac{\partial f_j}{\partial \overline{z}_k} \overline{\frac{\partial f_k}{\partial \overline{z}_j}}\, dV, \end{aligned} \tag{3.5}$$

where the final inequality uses for the first and only time that the domain G is pseudoconvex (which implies non-negativity of the boundary term).

Combining (3.3) and (3.5) shows that

$$\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2 \geq \sum_{j=1}^n \sum_{k=1}^n \int_G \left| \frac{\partial f_j}{\partial \overline{z}_k} \right|^2 dV.$$

Actually, the preceding inequality is not the one that is needed for the proof, but if you followed the calculation, then you should be able to keep track of some extra terms in the integration by parts to solve the following exercise.

Exercise 22. If a is an infinitely differentiable positive weight function, then

$$\begin{aligned} \int_G (|\overline{\partial} f|^2 + |\overline{\partial}^* f|^2)\, a\, dV &= \sum_{j=1}^n \sum_{k=1}^n \int_G \left| \frac{\partial f_j}{\partial \overline{z}_k} \right|^2 a\, dV + \int_{bG} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} f_j \overline{f}_k \, a\, dS \\ &\quad - \sum_{j=1}^n \sum_{k=1}^n \int_G \frac{\partial^2 a}{\partial z_j \partial \overline{z}_k} f_j \overline{f}_k \, dV + 2 \operatorname{Re} \Bigg\langle \sum_{k=1}^n f_k \frac{\partial a}{\partial z_k}, \overline{\partial}^* f \Bigg\rangle \\ &\geq - \sum_{j=1}^n \sum_{k=1}^n \int_G \frac{\partial^2 a}{\partial z_j \partial \overline{z}_k} f_j \overline{f}_k \, dV + 2 \operatorname{Re} \Bigg\langle \sum_{k=1}^n f_k \frac{\partial a}{\partial z_k}, \overline{\partial}^* f \Bigg\rangle. \end{aligned}$$

In the preceding exercise, replace the positive weight function a by $1 - e^b$, where b is a smooth negative function. Then

$$\frac{\partial^2 a}{\partial z_j \partial \overline{z}_k} = -e^b \frac{\partial^2 b}{\partial z_j \partial \overline{z}_k} - e^b \frac{\partial b}{\partial z_j} \frac{\partial b}{\partial \overline{z}_k},$$

so it follows that

$$\begin{aligned} &\int_G (|\overline{\partial} f|^2 + |\overline{\partial}^* f|^2)\, a\, dV \\ &\geq \sum_{j=1}^n \sum_{k=1}^n \int_G \frac{\partial^2 b}{\partial z_j \partial \overline{z}_k} f_j \overline{f}_k\, e^b\, dV + \int_G \left| \sum_{k=1}^n \frac{\partial b}{\partial z_k} f_k \right|^2 e^b\, dV - 2\operatorname{Re} \left\langle \sum_{k=1}^n f_k \frac{\partial b}{\partial z_k} e^{b/2}, e^{b/2}\, \overline{\partial}^* f \right\rangle. \end{aligned}$$

Applying the Cauchy-Schwarz inequality to the last term on the right-hand side and using that $a + e^b = 1$ shows that

$$\|\overline{\partial}^* f\|^2 + \|\overline{\partial} f\|^2 \geq \int_G |\overline{\partial}^* f|^2 + a|\overline{\partial} f|^2\, dV \geq \sum_{j=1}^n \sum_{k=1}^n \int_G \frac{\partial^2 b}{\partial z_j \partial \overline{z}_k} f_j \overline{f}_k e^b\, dV.$$

Now choose a point p in G, let δ denote the diameter of G, and set the negative function b equal to $-1 + |z - p|^2/\delta^2$. The preceding inequality then implies that

$$\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f|^2 \geq \|f\|^2/(\delta^2 e).$$

Thus the basic estimate (3.2) holds with the constant C equal to e times the square of the diameter of the domain G.