Fundamental Solution for $\Box_b$ on Quadrics

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Background

Let $\phi : \mathbb{C}^n \times \mathbb{C}^n \mapsto \mathbb{C}^m$ be a conjugate symmetric, bilinear map
A *Quadric* is a submanifold of the form

$$G = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m;\ \operatorname{Im} w = \phi(z, z)\}$$

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Let $t = \operatorname{Re}\{w\}$. Identify $(z, t + i\phi(z, z)) \in G$ with
$(z, t) \in \mathbb{C}^n \times \mathbb{R}^m$

Identification of Points on G

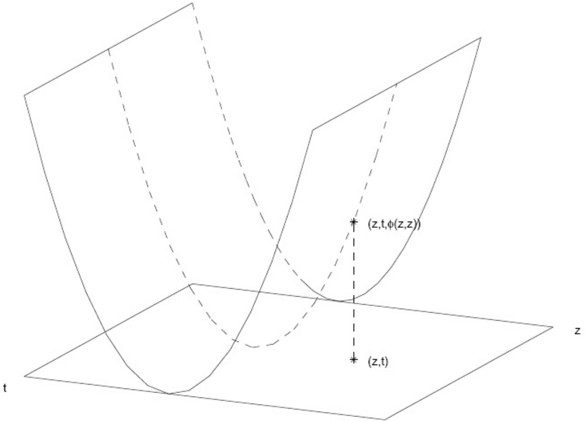

Figure: $(z, t + i\phi(z, z))$ identified with (z, t)

Background Cont.

Recall

$$G = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m;\ \operatorname{Im} w = \phi(z, z)\}$$

and $t = \operatorname{Re}\{w\}$.

G has a *Lie Group Structure*:

$$gg' = (z, w)(z', w') = \big(z + z', w + w' + 2i\phi(z', z)\big)$$

$$gg' = (z, t)(z', t') = \big(z + z', t + t' + 2 \operatorname{Im} \phi(z, z')\big)$$

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Let $v_1, \ldots, v_n$ be any orthonormal basis for $\mathbb{C}^n$. Then the (right) *invariant vector fields* are:

$$
\begin{aligned}
X_j &= \partial_{v_j} - 2\,\mathrm{Im}\,\phi(z, v_j) \cdot D_t,\ \ 1 \le j \le n \\
Y_j &= \partial_{Jv_j} + 2\,\mathrm{Re}\,\phi(z, v_j) \cdot D_t,\ \ 1 \le j \le n \\
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Tangential Cauchy-Riemann Vector Fields:

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Z_j &= \frac{1}{2}(X_j - iY_j) = \frac{\partial}{\partial z_j} - i\overline{\phi(z, v_j)} \cdot D_t \\
\overline{Z}_j &= \frac{1}{2}(X_j + iY_j) = \frac{\partial}{\partial \overline{z}_j} + i\phi(z, v_j) \cdot D_t
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Note: $[Z_\ell, \overline{Z}_j] = 2i\phi(v_\ell, v_j) \cdot D_t$. This is the *Levi form*.

$\Box_b$ Calculation (from Peloso/Ricci 2003)

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For any orthonormal basis, $v_1, \ldots, v_n$ for $\mathbb{C}^n$, and $|L| = q$:

$$\Box_b(f_L d\overline{z}^L) = \sum_{|I|=q} \Box_{LI}(f_L) d\overline{z}^I$$

where the diagonal terms are:

$$\Box_{LL} = \frac{-1}{4} \sum_{k=1}^{n} (X_k^2 + Y_k^2) + i \left(\sum_{k \in L} \phi(v_k, v_k) \cdot D_t - \sum_{k \notin L} \phi(v_k, v_k) \cdot D_t \right)$$

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For the off-diagonal terms, if $|I \cap L| = q - 1$, then

$$\Box_{LI} = \pm [Z_\ell, \overline{Z}_j] = \pm 2i\phi(v_\ell, v_j) \cdot D_t$$

where $\ell = L - I$ and $j = I - L$. If $|I \cap L| < q - 1$, then $\Box_{LI} = 0$.

Solvability of $\Box_b$

Fix $\lambda \in \mathbb{R}^m$. Let

$$\phi^\lambda(z,\zeta) = \phi(z,\zeta)\cdot\lambda$$

Theorem

(Peloso/Ricci 2003) Let $n^+(\lambda)$, resp $n^-(\lambda)$ be the number of positive, resp. negative, eigenvalues of ϕ^λ. Then $\Box_b$ is solvable on $(0,q)$-forms if and only if for every $\lambda \in R^m$, either $n^+(\lambda) \neq q$ or $n^-(\lambda) \neq n-q$.

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Goal. To find an explicit formula for the fundamental solution, N to $\Box_b$:

$$\Box_b N = I - P$$

where $P =$ orthog. projection of $L^2_{(0,q)}(G)$ onto the kernel of $\Box_b$.

Eigenvalue Notation

Fix $\lambda \in \mathbb{R}^m$, let $v_i = v_i^\lambda$, $i = 1, \ldots, n$ be an orthonormal basis for $\mathbb{C}^n$ which diagonalizes the form

$$\phi^\lambda(z, \zeta) = \phi(z, \zeta) \cdot \lambda$$

Let $\mu_i = \mu_i^\lambda$, $i = 1, \ldots, \nu^\lambda \leq n$ be the nonzero eigenvalues for ϕ^λ. For simplicity, assume $\nu^\lambda = n$.

Representation Theory for Quadrics

Define the group transform, for $f \in L^2(G)$, $\lambda \in \mathbb{R}^m$:

$$\tilde{f}_\lambda(\cdot) = \int_{(z,t)\in G} f(z,t)\pi_\lambda(z,t)(\cdot)\, dzdt$$

as an operator on $L^2(\mathbb{R}^n)$, where for $h \in L^2(\mathbb{R}^n)$,

$$\pi_\lambda(z = x + iy, t)(h)(\xi) = e^{i(\lambda\cdot t)} e^{-2i\sum_{j=1}^{n} \mu_j^\lambda y_j(\xi_j + x_j)} h(\xi + 2x)$$

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Under the transform, $X_j \to 2D_{\xi_j}$, $Y_j \to -2i\mu_j^\lambda \xi_j$, $D_{t_j} \to i\lambda_j$.

More Representation Theory

Recall

$$\Box_{LL} = \frac{-1}{4}\sum_{k=1}^{n}(X_k^2+Y_k^2)+i\left(\sum_{k\in L}\phi(v_k,v_k)\cdot D_t - \sum_{k\notin L}\phi(v_k,v_k)\cdot D_t\right)$$

More Representation Theory

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Since $X_j \to 2D_{\xi_j}$, $Y_j \to -2i\mu_j^\lambda \xi_j$, and $D_{t_j} \to i\lambda_j$,
$\Box_{LL}$ transforms to

$$\Box_{L,\xi}^\lambda = -\Delta_\xi + \sum_{k=1}^{n}(\mu^\lambda \xi_k)^2 + \sum_{k\in L}\mu_k^\lambda - \sum_{k\notin L}\mu_k^\lambda.$$

$\Box_{L,\xi}^\lambda$ is a multi-dimensional Hermite operator.

Fundamental Solution to $\Box_b$

So using the group transform, solving $\Box_b N = I - P$ becomes equivalent to solving

$$\Box^{\lambda}_{L,\xi} \tilde{N}_{\lambda} = I - P^{\lambda}_{L}$$

where P^{λ}_{L} is the orthogonal projection onto the zeroth eigenspace of the Hermite operator $\Box^{\lambda}_{L,\xi}$.

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Define the Hermite functions:

$$\psi_{\ell}(x) = c_{\ell} \frac{d^{\ell}}{dx^{\ell}} \{e^{-x^2}\} e^{x^2/2}$$

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A complete basis of eigenfunctions on $L^2(\mathbb{R}^n)$ for $\Box^{\lambda}_{L,\xi}$ is

$$\Psi^{\lambda}_{\ell}(\xi) = \prod_{j=1}^{n} \psi^{\lambda}_{\ell_j}(\xi_j).$$

where $\ell = (\ell_1, \ldots, \ell_n)$, $\ell_j \geq 0$, and

$$\psi^{\lambda}_{\ell_j}(\xi_j) = \psi_{\ell_j}(|\mu^{\lambda}_j|^{1/2}\xi_j)|\mu^{\lambda}_j|^{1/4}$$

Fundamental Solution to $\Box_b$

Eigenvalues:

$$\Box^{\lambda}_{L,\xi}\Psi^{\lambda}_{\ell} = \Lambda^{\lambda}_{\ell}\Psi^{\lambda}_{\ell}$$

where $\Lambda^{\lambda}_{\ell} = \sum_{j=1}^{n}(2\ell_j + 1)|\mu^{\lambda}_j| + \left(\sum_{j\in L}\mu^{\lambda}_j - \sum_{j\notin L}\mu^{\lambda}_j\right)$

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Theorem

(Peloso/Ricci, 2003) Suppose $|L| = q$. If the number of negative eigenvalues of ϕ^{λ} is not equal to q or if the number of positive eigenvalues of ϕ^{λ} is not equal to $n - q$, then the Kernel of the operator $\Box^{\lambda}_{L,\xi}$ is zero and its inverse is given by the operator

$$N^{\lambda} = \sum_{\ell} \frac{1}{\Lambda^{\lambda}_{\ell}} P^{\lambda}_{\ell}$$

where P^{λ}_{ℓ} = orthogonal proj. onto the space spanned by $\Psi^{\lambda}_{\ell}(\xi)$.

Fundamental Solution to $\Box_b$

Now in certain cases, we can evaluate

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in closed form and then evaluate its inverse group transform. This process involves manipulating Mehler's formula and lots of inverse Fourier transform calculations.

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Special Case: Take the case when $|\mu_j^\lambda| = |\lambda|$, all $1 \leq j \leq n$. This includes the case of the Heisenberg group. It also includes the following codimension two example:

$$\{(w_1, w_2, z_1, z_2) \in C^4; \;\; \text{Im}\, w_1 = 2\,\text{Re}\{z_1 \bar{z}_2\}, \; \text{Im}\, w_2 = |z_1|^2 - |z_2|^2\}$$

where $\mu_1^\lambda = |\lambda|$ and $\mu_2^\lambda = -|\lambda|$.

Special Case: $|\mu_j^\lambda| = |\lambda|$

When $|\mu_j^\lambda| = |\lambda|$,

$$\begin{aligned}\Lambda_\ell^\lambda &= \sum_{j=1}^{n}(2\ell_j+1)|\mu_j^\lambda| + \left(\sum_{j\in L}\mu_j^\lambda - \sum_{j\notin L}\mu_j^\lambda\right) \\ &= 2(\sum_{j=1}^{n}\ell_j + J)|\lambda| = 2(|\ell| + J)|\lambda|\end{aligned}$$

where J is a positive integer (depending on L). Obtain:

$$N(z,\hat{\lambda}) = C_n|\lambda|^{n-1}\int_0^\infty s^{J-1}(s+2)^{n-J-1}e^{-(s+1)|\lambda||z|^2}\,ds$$

One can recover the usual formulas for N for the Heisenberg group.

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One can recover the usual formulas for N for the Heisenberg group. For the codimension two example in C^4, $q=0$, then

$$N(z,t) = \frac{C}{(|t|^2+|z|^4)^{3/2}}.$$

Fundamental Solution to $\Box_b$

When $|\mu_j^\lambda|$ are not all equal

Example. Take the hypersurface in C^3,

$$\{(w, z_1, z_2);\ \mathrm{Im}\, w = \sigma_1 |z_1|^2 + \sigma_2 |z_2|^2\}$$

$\sigma_j > 0$. Take the case $L = (1, 0)$ $(q = 1)$. Obtain:

$$\begin{aligned} N(z,t) &= C \int_0^1 \frac{P_1(r,\sigma)\sigma_1\sigma_2\ dr}{(it + s_1(r)\sigma_1|z_1|^2 + s_2(r)\sigma_2|z_2|^2)^2} \\ &+ C \int_0^1 \frac{P_2(r,\sigma)\sigma_1\sigma_2\ dr}{(-it + s_1(r)\sigma_1|z_1|^2 + s_2(r)\sigma_2|z_2|^2)^2} \end{aligned}$$

where C is the same constant for both integrals, and where

$$P_j = \frac{r^{\sigma_j - 1}\, dr}{(1 - r^{\sigma_1})(1 - r^{\sigma_2})} \quad s_j(r) = \frac{1 + r^{\sigma_j}}{1 - r^{\sigma_j}}$$