Math 222 - - Exam I Solutions

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1. We wish to write the vector

$$w = \begin{pmatrix} -6 \\ 10 \\ -19 \end{pmatrix}$$

as a linear combination of $v_1, \ v_2, \ v_3$. This means we must hunt for constants $x_1, \ x_2 \ x_3$ with $w = x_1v_1 + x_2v_2 + x_3v_3$. This is a system of three equations and three unknowns whose augmented matrix is

$$A := \begin{bmatrix} 1 & -2 & 2 & | & -6 \\ -2 & 6 & 0 & | & 10 \\ 3 & -2 & 11 & | & -19 \end{bmatrix}$$

We use Gaussian elimination to solve this system. First we add 2 times the first row to the second:

$$\begin{bmatrix} 1 & -2 & 2 & -6 \\ 0 & 2 & 4 & -2 \\ 3 & -2 & 11 & -19 \end{bmatrix}$$

Then we add -3 times the first row to the third:

$$\begin{bmatrix} 1 & -2 & 2 & -6 \\ 0 & 2 & 4 & -2 \\ 0 & 4 & 5 & -1 \end{bmatrix}$$

Next we divide the second row by 2:

$$\begin{bmatrix} 1 & -2 & 2 & -6 \\ 0 & 1 & 2 & -1 \\ 0 & 4 & 5 & -1 \end{bmatrix}$$

Then we add -4 times the second row to the third:

$$\begin{bmatrix} 1 & -2 & 2 & -6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

Back substitution now yields $x_3 = -1,\ x_2 = 1,\ x_1 = -2$ and we obtain $W = -2v_1 + v_2 - v_3$. This linear combination is unique since the row echelon form of the matrix does not have a row of zeros and therefore no free variables.

2. The matrix B is obtained from A by adding $-2/3$ times the first row to the third row. The elementary matrix E which has this affect is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-2}{3} & 0 & 1 \end{bmatrix}$$

3. The nullspace of a matrix A is the set of vectors x with $Ax = 0$. In our case, we are to examine the nullspace of

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which means we must examine the following augmented matrix

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 4 & |0 \\ 0 & 0 & 1 & -2 & -5 & |0 \\ 0 & 0 & 0 & 0 & 1 & |0 \end{bmatrix}$$

This matrix is already in row echelon form. Columns 1, 3 and 5 are the pivots (with leading 1s) and columns 2 and 4 are free. Thus x_2 and x_4 can be given arbitrary parameters. Since there are two of them, the dimension of the nullspace is two.

4. We are given the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ -3 & 1 & 2 \end{bmatrix}$$

We are to use row operations to reduce this matrix to an upper triangular matrix and compute the determinant along the way. We use the following facts. A row switch changes the determinant by a minus sign. Adding a multiple of a row to another does not change the determinant. Multiplying a row by a factor changes the determinant by that factor.

First we switch the first and second rows.

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

Then we add (-2) times the first row to the second:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ -3 & 1 & 2 \end{bmatrix}$$

Then we add 3 times the first row to the third:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & 5 \end{bmatrix}$$

Finally, we add -1 times the second row to the third:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 8 \end{bmatrix}$$

This matrix is upper triangular. Its determinant is 8. We used one row switch which changes the determinant by a minus sign. The rest of the row operations involve adding multiples of rows to other rows and these operations do not change the determinant. Therefore the determinant of the original matrix is -8.

5. Part a) The set

$$S = \{(x, y, z) \in R^3; x - 2y + z = 0\}$$

IS a vector space. For scalar multiplication: if $(x, y, z) \in S$ and r is a number, then we must show $(rx, ry, rz) \in S$. We have $rx - 2ry + rz = r(x-2y+z) = 0$ and thereforem $(rx, ry, rz) \in S$. For addition, suppose

$(x_1, y_1, z_1),\ (x_2, y_2, z_2) \in S$. We must show $(x_1+x_2, y_1+y_2, z_1+z_2) \in S$. We have

$$\begin{aligned} (x_1+x_2) - 2(y_1+y_2) + (z_1+z_2) &= (x_1 - 2y_1 + z_1) + (x_2 - 2y_2 + z_2) \\ &= 0 + 0 \end{aligned}$$

Part b) The set

$$S = \{x^2 - 2y^2 + z^2 = 0\}$$

is NOT a subspace. It is closed under scalar multiplication but not addition. For example $p = (1, 1, 1)$ and $q = (\sqrt(2), 1, 0)$ both belong to S, but their sum, $p + q$ does not.

Part c) Since the vectors $(1, 2)$ and $(-1, 3)$ are not colinear, the span of these two vectors is all of R^2, which is certainly a subspace of itself (closed under addition and scalar multiplication). So this space IS a subspace.

Part d) The set of polynomials of degree 2 with nonnegative coefficients is NOT a subspace because it is not closed under scalar multiplication by negative numbers.

Part e) The set, S, of all differentiable functions, f, with $f'(0) = 0$ IS a subspace. If f belongs to S and r is a scalar, then $(rf)'(0) = r(f'(0)) = r0 = 0$. So S is closed under scalar multiplication. If f and g belong to S, then $(f+g)'(0) = f'(0) + g'(0) = 0 + 0 = 0$. So S is closed under addition.

6. We are to show that a matrix A is nonsingular if and only if its nullspace is $\{0\}$.

   First we assume that A is nonsingular, and we prove its nullspace is $\{0\}$. If A is nonsingular, then its inverse exists. If x is in the nullspace of A, then $Ax = 0$. Multiplying this equation by A^{-1} we obtain

   $$\begin{aligned} A^{-1}Ax &= A^{-1}0 \\ \text{or } x &= 0 \end{aligned}$$

   Thus x has to be the zero vector, and so the nullspace is $\{0\}$.

   Now we assume the nullspace consists of only the zero vector. We must show A^{-1} exists. Since the equation $Ax = 0$ only has zero in its solution, the row echelon form of the augmented matrix $(A|0)$ is

$$\left[\begin{array}{cccc|c} 1 & * & \dots & * & 0 \\ 0 & 1 & \dots & * & 0 \\ 0 & 0 & \ddots & * & 0 \\ 0 & 0 & \dots & 1 & 0 \end{array}\right]$$

We can further use the ones in the pivot locations to knock out the entries above the diagonal. Thus its *reduced row echelon form* is the identity matrix. Since each row operation is equivalent to multiplying by an elementary matrix, we see that there is a series of elementary matrices $E_1, \dots, E_k$ with

$$E_1 \dots E_k A = I$$

Clearly, the inverse of A is the matrix $E_1 \dots E_k$.

7. We are to show that for a nonsingular matrix A, $\det A^{-1} = 1/\det A$. Taking determinants of both sides of the equation $AA^{-1} = I$, we obtain

$$\begin{aligned} \det(AA^{-1}) &= \det I \\ \det A \det A^{-1} &= 1 \end{aligned}$$

Dividing both sides by $\det A$, we obtain

$$\det A^{-1} = 1/\det A$$

as desired.