Math 222 - - Exam II

Instructor - Al Boggess

Fall 1998

1. Let

$$u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{and} \quad v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Suppose x is the vector $x = u_1 - 2u_2$ in R^2. Express x in terms of the basis v_1 and v_2.

SOLUTION. The transition matrix to change from the $u-$ basis to the $v-$basis is

$$S = V^{-1}U$$

where U is the matrix whose columns are the us:

$$U := \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

and V is the matrix whose columns are the vs:

$$V := \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{aligned} S &= V^{-1}U \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} \end{aligned}$$

The vector $x = u_1 - 2u_2$ transforms to the vector $av_1 + bv_2$ where

$$\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix} &= (S)\begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -7 \end{pmatrix} \end{aligned}$$

Thus the answer is $3v_1 - 7v_2$.

2. Let

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as in Problem 1. Let $L : R^2 \mapsto R^2$ be the linear transformation given by $L(x) = (A)(x)$ where

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

Find the matrix of L with respect to the basis v_1, v_2.

SOLUTION. The matrix which represents L with respect to the basis v_1, v_2 is

$$\begin{aligned} B &= V^{-1}AV \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 1 \\ -3 & 1 \end{bmatrix} \end{aligned}$$

as the answer.

3. Find a basis for the kernel and range of the linear map $L : R^4 \mapsto R^3$, $L(x) = (A)(x)$ where

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 1 & -1 \\ -3 & 1 & 2 & -3 \end{pmatrix}$$

SOLUTION.

The row reduced form of the given matrix is

$$RREF(A) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This indicates that the first two columns of the original matrix are linearly independent and span the column space. So a basis for the column space is

$$v_1 = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

A basis for A can be found by looking at the row reduced form of A:

$$RREF(A) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The variables x_3 and x_4 are free. Letting $x_3 = 1$ and $x_4 = 0$ leads to the vector $N_1 = (-1, -1, 1, 0)^T$. Letting $x_3 = 0$ and $x_4 = 1$ leads to the vector $N_2 = (2, 3, 0, 1)^T$. The vectors N_1 and N_2 form a basis for the nullspace of A.

4. Find a basis for the orthogonal complement of the space in R^4 spanned by

$$u_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

SOLUTION. First form the matrix whose rows are u_1 and u_2:

$$U = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

The orthogonal complement of the space spanned by u_1 and u_2 is the same as the kernel of this matrix. The reduced form of this matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1/2 & 1 \end{bmatrix}$$

The variables x_3 and x_4 are free. Letting $x_3 = 1$ and $x_4 = 0$ leads to the vector $N_1 = (-1, -1/2, 1, 0)^T$. Letting $x_3 = 0$ and $x_4 = 1$ leads to the vector $N_2 = (-1, 0, 0, 1)^T$. The vectors N_1 and N_2 form a basis for the nullspace of U and therefore the orthogonal complement of the space spanned by u_1 and u_2.

5. Let P_n be the vector space of polynomials of degree less than n. Let

$$v_1 = 1, \quad v_2 = 1 + x, \quad v_3 = x + x^2, \quad v_4 = x^2 + x^3$$

be a basis for P_3. Let

$$u_1 = 1, \quad u_2 = 1 + x, \quad u_3 = x + x^2$$

be a basis for P_2. Let $D : P_3 \mapsto P_2$ be the map $D(p) = p'$ for $p \in P_3$. Show that D is a linear transformation. Compute the matrix of D with respect to the basis u_i, $1 \leq i \leq 4$ and v_j, $1 \leq j \leq 3$.

SOLUTION. D is the differentiation operator and we must show that D is linear. We have

$$\begin{aligned} D\{\alpha p_1 + \beta p_2\} &= \{\alpha p_1 + \beta p_2\}' \\ &= \alpha p_1' + \beta p_2' \\ &= \alpha D(p_1) + \beta D(p_2) \end{aligned}$$

So D is linear.

To compute the matrix of D with respect to the us and vs, we compute $D(u_i)$, for $1 \leq i \leq 4$ and express them in terms of the vs. We have

$$\begin{aligned} D(u_1) &= D(1) = 0 \\ D(u_2) &= D(1 + x) = 1 = v_1 \\ D(u_3) &= D(x + x^2) = 1 + 2x = -v_1 + 2v_2 \\ D(u_4) &= D(x^2 + x^3) = 2x + 3x^2 = v_1 - v_2 + 3v_3 \end{aligned}$$

The coefficients form the columns of the matrix. So the matrix for D with respect to the us and vs is

$$\begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

6. Suppose $v_1,\ v_2$ is a basis for R^2 and let L_1 and L_2 are two linear transformations from R^2 to R^2. Show that if $L_1(v_i) = L_2(v_i)$ for $i = 1, 2$, then $L_1(v) = L_2(v)$ for all $v \in R^2$.

   SOLUTION. Since v_1 and v_2 form a basis of R^2, any v can be expressed as $v = \alpha v_1 + \beta v_2$. So

$$\begin{array}{ccc} L_1(v) & = & L_1(\alpha v_1 + \beta v_2) \\ & =^{\alpha} L_1(v_1) + \beta L_1(v_2) & \end{array}$$

   Since L_1 and L_2 agree on v_1 and v_2, we have

$$\begin{aligned} L_1(v) &= \alpha L_1(v_1) + \beta L_1(v_2) \\ &= \alpha L_2(v_1) + \beta L_2(v_2) \\ &= L_2(\alpha(v_1 + \beta v_2) \\ &= L_2(v) \end{aligned}$$

   Thus, $L_1(v) = L_2(v)$.

7. Suppose $u_1, \ldots, u_k$ is a basis for the subspace $U \subset R^n$. Show that a vector $v \in R^n$ belongs to $U^\perp$ (the orthogonal complement of U) if and only if $v \cdot u_i = 0$ for each i with $1 \le i \le k$.

   SOLUTION. Suppose $v \cdot u_i = 0$ for $1 \le i \le k$. We must show that v belongs to $U^\perp$, or equivalently, that $v \cdot u = 0$ for all $u \in U$. Now if u is any vector in U, then u can be expressed in terms of the basis:

$$u = \alpha_1 u_1 + \ldots + \alpha_k u_k$$

   Therefore

$$\begin{aligned} v \cdot u &= v \cdot (\alpha_1 u_1 + \ldots + \alpha_k u_k) \\ &= \alpha_1 (v \cdot u_1) + \ldots + \alpha_k (v \cdot u_k) \\ &= 0 \end{aligned}$$

   since $v \cdot u_i = 0$ for each i.

   Conversely, suppose $v \in U^\perp$, then $v \cdot u$ for all $u \in U$. Since each u_i belongs to U, clearly $v \cdot u_i = 0$ for each $1 \le i \le k$.