Math 414 - Solutions to Chapter 4

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1. The given function

$$f(x) = \begin{cases} -1 & 0 \le x < 1/4 \\ 4 & 1/4 \le x < 1/2 \\ 2 & 1/2 \le x < 3/4 \\ -3 & 3/4 \le x < 1 \end{cases}$$

can be written as

$$f(x) = -\phi(4x) + 4\phi(4x-1) + 2\phi(4x-2) - 3\phi(4x-3)$$

Using the equations

$$\begin{aligned} \phi(4x) &= (\psi(2x) + \phi(2x))/2 \\ \phi(4x-1) &= (\phi(2x) - \psi(2x))/2 \\ \phi(4x-2) &= \phi(4(x-1/2)) = (\psi(2(x-1/2)) + \phi(2(x-1/2)))/2 \\ \phi(4x-3) &= \phi(4(x-1/2)-1) = (\phi(2(x-1/2) - \psi(2(x-1/2)))/2 \end{aligned}$$

into the terms for $f(x)$ and collecting terms, we have

$$f(x) = 3/2\phi(2x) - 1/2\phi(2x-1) - 3/2\psi(2x) + 5/2\psi(2x-1)$$

The terms involving $\psi(2x)$ and $\psi(2x-1)$ belong to W_1. We need to further decompose the terms involving $\phi(2x)$ and $\phi(2x-1)$. The key equations are

$$\begin{aligned} \phi(2x) &= (\phi(x) + \psi(x))/2 \\ \phi(2x-1) &= (\phi(x) - \psi(x))/2 \end{aligned}$$

Substituting these equations and collecting terms gives

$$f(x) = -\phi(x) - (1/2)\psi(x) - 3/2\psi(2x) + 5/2\psi(2x-1)$$

The term involving $\phi(x)$ is the V_0 component of f. The term involving $\psi(x)$ is the W_0 component of f. The sum of the terms involving $\psi(2x)$ and $\psi(2x-1)$ is the W_1 component of f.

2. The dimension of a vector space is the number of basis elements. So suppose the dimension of A is n and the dimension of B is m. Let $a_1, \ldots a_n$ is a basis for A and $b_1, \ldots b_m$ is a basis for B. We would like to show that the collection $a_1, \ldots a_n, b_1, \ldots b_m$ is a basis for $A \oplus B$.

   First we show this collection spans. An arbitrary element of $A \oplus B$ is of the form $a + b$ with $a \in A$ and $b \in B$. Since $a_1, \ldots a_n$ is a basis for A, there are constants $r_1, \ldots, r_n$ with

$$a = r_1 a_1 + \ldots + r_n a_n.$$

   Likewise, there are constants $s_1, \ldots s_m$ with

$$b = s_1 b_1 + \ldots s_m b_m.$$

   Adding these two equations yields

$$a + b = r_1 a_1 + \ldots + r_n a_n + s_1 b_1 + \ldots s_m b_m$$

   which shows this collection spans.

   To show linear independence, suppose that

$$r_1 a_1 + \ldots + r_n a_n + s_1 b_1 + \ldots s_m b_m = 0$$

   We wish to show that all the coefficients (the rs and the ss) are zero. Let $a = r_1 a_1 + \ldots + r_n a_n \in A$ and let $b = s_1 b_1 + \ldots s_m b_m \in B$. Thus the above equation becomes $a + b = 0$. Taking the inner product of both sides with this equation with b gives

$$\langle (a+b), b \rangle = \langle 0, b \rangle = 0.$$

   Since $\langle a, b \rangle = 0$ (by hypothesis), we conclude that $\langle b, b \rangle = 0$, which means that $s_1 b_1 + \ldots s_m b_m = b = 0$. Since $b_1, \ldots, b_m$ is a basis for B, they are linearly independent. So all the s-coefficients must be zero. In the same manner (taking the inner product with a), we can show that $a = 0$ and hence all the r-coefficients must be zero.

   If A and B are not orthogonal, they may have a nonempty intersection. In this case, the dimensions are related by

$$dim(A + B) = dim(A) + dim(B) - dim(A \cap B).$$

In fact, a basis for $A+B$ can be constructed as follows. Start with a basis for $A \cap B$, say $c_1, \ldots c_k$ (assuming the dimension of $A \cap B$ is k). Expand this collection to a basis for A as $c_1, \ldots, c_, a_{k+1}, \ldots, a_n$ Also expand the collection of cs to a basis for B as $c_1, \ldots c_k, b_{k+1}, \ldots b_m$. Then a basis for $A+B$ can be shown to be

$$c_1, \ldots, c_k, a_{k+1}, \ldots, a_n, b_{k+1}, \ldots b_m.$$

which is a collection of $n+m-k$ elements.

3. V_0 is spanned only by $\phi(x)$ since any translate of ϕ is zero on the interval $0 \leq x \leq 1$. Thus, V_0 has dimension 1. Likewise, V_1 is spanned by $\phi(2x)$ and $\phi(2x-1)$. All other translates of $\phi(2x)$ are zero on the interval $0 \leq x \leq 1$. In general, the collection

$$\phi(2^n x - k), \quad k = 0, 1, 2, \ldots 2^n - 1$$

is a basis for V_n. All other translates of $\phi(2^n x)$ are zero on the interval $0 \leq x \leq 1$. Therefore the dimension of V_n is 2^n.

The same argument with ϕ replaced by ψ shows that the dimension of W_n is also 2^n.

For the second part, we use the decomposition

$$V_n = W_{n-1} \oplus W_{n-2} \oplus \ldots \oplus W_0 \oplus V_0$$

The dimension of the right side (using the previous problem) is

$$2^{n-1} + 2^{n-2} + \ldots 2^0 + 2^0$$

From the formula for the geometric series

$$2^{n-1} + 2^{n-2} + \ldots 2^0 = \frac{1-2^n}{1-2} = 2^n - 1$$

Therefore

$$(2^{n-1} + 2^{n-2} + \ldots 2^0) + 2^0 = (2^n - 1) + 1 = 2^n$$

which agrees with the dimension of V_n computed in the first part of the problem.

4. Suppose $f = \sum_k a_k \phi(2x - k)$ is orthogonal to V_0 which is spanned by the $\phi(x - l)$. We wish to show $a_0 = -a_1$, $a_2 = -a_3$ .... Now f is orthogonal to $\phi(x - l)$ for all l. Suppose $l = 0$, then since $\phi(x)$ is one for $0 \leq x \leq 1$ and zero otherwise, we have

$$0 = \langle f, \phi(x) \rangle = \int_0^1 \sum_k a_k \phi(2x - k)$$

The only values of k which contribute to this sum are $k = 0$ and 1. Any other translate of $\phi(2x)$ is zero on the interval $0 \leq x \leq 1$. Therefore the above equation becomes

$$0 = a_0 \int_0^1 \phi(2x) + a_1 \int_0^1 \phi(2x - 1)$$

Since the graphs of $\phi(2x)$ and $\phi(2x - 1)$ are boxes of width 1/2 and height 1, both integrals are 1/2. Therefore the above equation becomes

$$0 = a_0/2 + a_1/2$$

which implies that $a_0 = -a_1$, as desired.

In general, f is orthogonal to $\phi(x - l)$ for any l and so the above arguments can be repeated over the interval $l \leq x \leq l + 1$ and we conclude that $a_{2l} = -a_{2l+1}$.

5. First we claim that if $|f'(x)| \leq M$ then

$$|f(x) - f(y)| \leq M|x - y|$$

This follows from the Mean Value Theorem, which states

$$f(x) - f(y) = f'(c)(x - y)$$

for some c between x and y. Since $|f'(c)| \leq M$, the previous inequality now follows. So on the interval $k/2^n \leq x \leq (k+1)/2^n$ (which has width $1/2^n$), the function f varies by no more than $M/2^n$. The function $f_n(x) = \sum_k a_k \phi(2^n x - k)$ with $a_k = f(k/2^n)$ has constant value $f(k/2^k)$ on the interval $k/2^n \leq x \leq (k+1)/2^n$. Thus, f_n and f differ by no more than $M/2^n$. To make this quantity less than ϵ, we must require

$$M/2^n \leq \epsilon$$

or

$$M/\epsilon \leq 2^n$$

or

$$\log_2(M/\epsilon) \leq n$$

as claimed.

6. Since $V_j = V_{j-1} \oplus W_{j-1}$, the function $\phi(2^j x - k) \in V_j$ can be written as:

$$\phi(2^j x - k) = \sum_{l \in Z} \alpha_l \phi_{j-1,l}(x) + \beta_l \psi_{j-1,l}(x) \tag{1}$$

for some choice of constants α_l and β_l. Our goal is to compute β_l. Since $\phi_{j-1,l}$ and $\psi_{j-1,l}$ are orthonormal, we have

$$\begin{aligned} \beta_l &= \langle \phi(2^j x - k), \psi_{j-1,l} \rangle_{L^2} \\ &= 2^{(j-1)/2} \int_{-\infty}^{\infty} \phi(2^j x - k) \overline{\psi(2^{j-1} x - l)}\, dx. \end{aligned}$$

Now we use the defining equation for ψ:

$$\psi(x) = \sum_{k'} (-1)^{k'} \overline{p}_{1-k'} \phi(2x - k')$$

with x replaced by $2^{j-1}x - l$. We obtain

$$\beta_l = 2^{(j-1)/2} \int_{-\infty}^{\infty} \phi(2^j x - k) \sum_{k'} (-1)^{k'} p_{1-k'} \overline{\phi(2^j x - 2l - k')}\, dx.$$

Since $\{2^{j/2}\phi(2^j x - k), k \in Z\}$ is an orthonormal set, the only contributing term on the right occurs when $2l + k' = k$ or $k' = k - 2l$ and we obtain

$$\beta_l = 2^{-(j+1)/2} (-1)^k p_{1-k+2l} \quad * \tag{2}$$

So

$$\begin{aligned} \beta_l \psi_{j-1,l}(x) &= \beta_l 2^{(j-1)/2} \psi(2^{j-1}x - l) \quad \text{by definition of } \psi_{j-1,l} \\ &= 2^{-1}(-1)^k p_{1-k+2l} \psi(2^{j-1}x - l) \quad \text{by } (*). \end{aligned}$$

Thus from (1), the projection of $\phi(2^j x - k)$ onto W_{j-1} is

$$\sum_{l \in Z} \beta_l \psi_{j-1,l} = \sum_{l \in Z} 2^{-1} p_{1-k+2l} \psi(2^{j-1}x - l).$$

8. The orthogonality condition can be stated as

$$\int \psi(x-k)\overline{\phi(x-l)}\, dx = \delta_{kl} = 0$$

By replacing $x-l$ by x in the above integral and relabeling $n = k - l$, this orthogonality condition can be re-stated as

$$\int \psi(x-n)\overline{\phi(x)}\, dx = 0. \tag{3}$$

Recall that Plancherel's identity for the Fourier transform states

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi \qquad \text{for } f,\ g \in L^2.$$

We apply this identity with $f(x) = \psi(x-n)$ and $g(x) = \phi(x)$. As shown in class, $\widehat{\psi(x-n)}(\xi) = \widehat{\psi}(\xi)e^{-in\xi}$. So the orthogonality condition (3) can be re-stated as

$$\int_{-\infty}^{\infty} \widehat{\psi}(\xi)\overline{\widehat{\phi}(\xi)} e^{-in\xi}\, d\xi = 0$$

By dividing up the real line into the intervals $I_j = [2\pi j, 2\pi(j+1)]$ for $j \in Z$ this equation can be written as

$$\sum_{j \in Z} \int_{2\pi j}^{2\pi(j+1)} \hat{\phi}(\xi)\overline{\hat{\psi}(\xi)} e^{-in\xi}\, d\xi = .$$

Now replace ξ by $\xi + 2\pi j$. The limits of integration change to 0 and 2π:

$$\int_0^{2\pi} \sum_{j \in Z} \hat{\phi}(\xi + 2\pi j)\overline{\hat{\psi}(\xi + 2\pi j)} e^{-in(\xi + 2\pi j)}\, d\xi = 0.$$

Since $e^{2\pi i j} = 1$, for $j \in Z$, this equation becomes

$$\int_0^{2\pi} \sum_{j \in Z} \hat{\phi}(\xi + 2\pi j)\overline{\hat{\psi}(\xi + 2\pi j)} e^{-in(\xi)}\, d\xi = 0. \tag{4}$$

Let

$$F(\xi) = 2\pi \sum_{j \in Z} \hat{\phi}(\xi + 2\pi j)\overline{\hat{\psi}(\xi + 2\pi j)}.$$

The orthogonality condition (4) becomes

$$(1/2\pi) \int_0^{2\pi} F(\xi) e^{-in\xi}\, d\xi = 0 \tag{5}$$

The function F is 2π - periodic because

$$\begin{aligned} F(\xi+2\pi) &= 2\pi \sum_{j\in Z} \hat{\phi}(\xi+2\pi(j+1))\overline{\hat{\psi}(\xi+2\pi(j+1))} \\ &= 2\pi \sum_{j'} \hat{\phi}(\xi+2\pi j')\overline{\hat{\psi}(\xi+2\pi j')} \end{aligned}$$

where the last equality uses the change of index $j' = j+1$. Since F is periodic, it has a Fourier series, $\sum \alpha_n e^{inx}$, where the Fourier coefficients are given by $\alpha_n = (1/2\pi)\int_0^{2\pi} F(\xi)e^{-in\xi}\,d\xi$. Thus, the orthonormality condition, (5), is equivalent to $\alpha_n = 0$, which in turn is equivalent to the statement $F(\xi) = 0$. This completes the proof.