Solutions to Chapter 2

1. The following Maple code computes the Fourier series of $f(x) = x^2$

```
>  f:=x->x^2;
```

$$f := x \to x^2$$

```
>  (1/(2*Pi))*Int(f(x),x=-Pi..Pi); a0:=value("); #the coefficient
of a_0
```

$$\frac{1}{2}\,\frac{\int_{-\pi}^{\pi} x^2\,dx}{\pi}$$

$$a0 := \frac{1}{3}\,\pi^2$$

```
>  a:=n->value( (1/Pi)*Int(f(x)*cos(n*x),x=-Pi..Pi) ); # the value
of a_n
```

$$a := n \to \mathrm{value}\left(\frac{\int_{-\pi}^{\pi} \mathrm{f}(x)\cos(n\,x)\,dx}{\pi}\right)$$

```
>  b:=n->value( (1/Pi)*Int(f(x)*sin(n*x),x=-Pi..Pi) ); #the value
of b_n
```

$$b := n \to \mathrm{value}\left(\frac{\int_{-\pi}^{\pi} \mathrm{f}(x)\sin(n\,x)\,dx}{\pi}\right)$$

The following is the partial Nth sum for the Fourier series

$$0$$

```
>  S:=N->a0+value(Sum(a(n)*cos(n*x)+b(n)*sin(n*x ),n=1..N));
```

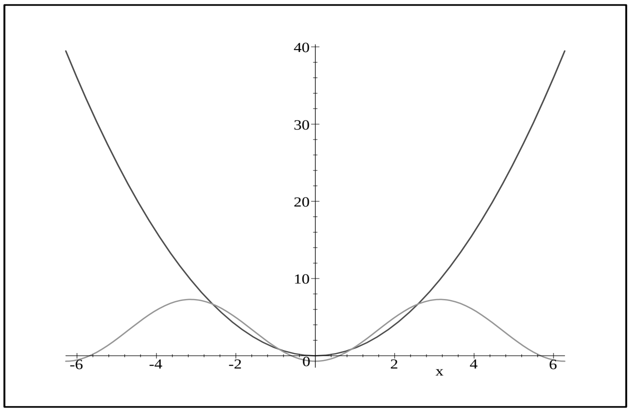
$$S := N \to a0 + \mathrm{value}\left(\sum_{n=1}^{N} (\mathrm{a}(n)\cos(n\,x) + \mathrm{b}(n)\sin(n\,x))\right)$$

```
>  S(7); #the first 7 terms of the Fourier series
```

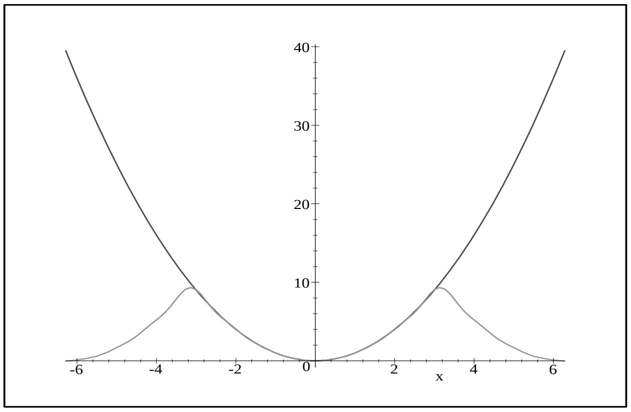
$$\frac{1}{3}\,\pi^2 - 4\cos(x) + \cos(2\,x) - \frac{4}{9}\cos(3\,x) + \frac{1}{4}\cos(4\,x) - \frac{4}{25}\cos(5\,x) + \frac{1}{9}\cos(6\,x) - \frac{4}{49}\cos(7\,x)$$

Note that there are only cosine terms since f(x) is even.

```
>  plot({f(x), S(1)},x=-2*Pi..2*Pi); #plot of S(1)
```



```
>  plot({f(x), S(7)},x=-2*Pi..2*Pi); #plot of S(7)
>  the first seven terms of the Fourier series
```



Note the accuracy increase from N=1 to N=7. Of course, the Fourier series only approximates on the inteveral [-Pi, Pi].

Fourier series for the interval [-A..A]

Here are the same Maple commands for the interval [-A,A] for the same function $f(x) = x^2$ (but can also be applied to any other function f).

```
>  f:=x->x^2;
```

$$f := x \to x^2$$

```
>  (1/(2*A))*Int(f(x),x=-A..A); a0:=value("); #the coefficient of
a_0
```

$$\frac{1}{2}\,\frac{\int_{-A}^{A} x^2\,dx}{A}$$

$$a0 := \frac{1}{3}\,A^2$$

```
>  a:=n->value( (1/A)*Int(f(x)*cos(n*Pi*x/A),x=-A..A) ); # the value
of a_n
```

$$a := n \to \mathrm{value}\left(\frac{\int_{-A}^{A} \mathrm{f}(x)\cos(\frac{n\,\pi\,x}{A})\,dx}{A}\right)$$

```
>  b:=n->value( (1/A)*Int(f(x)*sin(n*Pi*x/A),x=-A..A) ); #the value
of b_n
```

$$b := n \to \mathrm{value}\left(\frac{\int_{-A}^{A} \mathrm{f}(x)\sin(\frac{n\,\pi\,x}{A})\,dx}{A}\right)$$

The following is the partial Nth sum of the Fourier series

```
>  S:=N->a0+value(Sum(a(n)*cos(n*Pi*x/A)+b(n)*si n(n*Pi*x/A),n=1..N));
```

$$S := N \to a0 + \mathrm{value}\left(\sum_{n=1}^{N} (\mathrm{a}(n)\cos(\frac{n\,\pi\,x}{A}) + \mathrm{b}(n)\sin(\frac{n\,\pi\,x}{A}))\right)$$

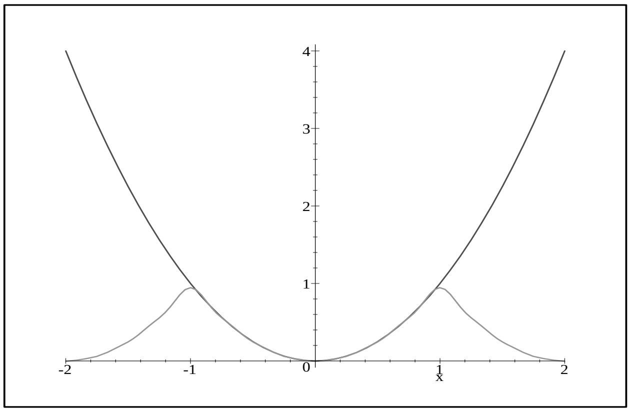
```
>  A:=1; #Set A to 1
```

$$A := 1$$

```
>  S(7);
```

$$\frac{1}{3} - 4\,\frac{\cos(\pi\,x)}{\pi^2} + \frac{\cos(2\,\pi\,x)}{\pi^2} - \frac{4}{9}\,\frac{\cos(3\,\pi\,x)}{\pi^2} + \frac{1}{4}\,\frac{\cos(4\,\pi\,x)}{\pi^2} - \frac{4}{25}\,\frac{\cos(5\,\pi\,x)}{\pi^2} + \frac{1}{9}\,\frac{\cos(6\,\pi\,x)}{\pi^2}$$
$$- \frac{4}{49}\,\frac{\cos(7\,\pi\,x)}{\pi^2}$$

```
>  plot({f(x), S(7)},x=-2..2); #plot of S(7) x=-1..1
```



Cosine and Sine expansions

2. The following Maple code will generate the cosine expansion of $f(x)$ on $0 \leq x \leq A$. Of course, since $f(x) = x^2$ is even, only cosine terms appear in its expansion on the interval $-A \leq x \leq A$. So the cosine expansion on $0 \leq x \leq A$ is the same as the expansion given in problem 1. However, the Maple code for a general f is given below.

```
>  f:=x->x^2;
```

$$f := x \to x^2$$

```
>  (1/(A))*Int(f(x),x=0..A); a0:=value("); #the coefficient of a_0
```

$$\int_0^1 x^2\,dx$$

$$a0 := \frac{1}{3}$$

```
>  a:=n->value( (2/A)*Int(f(x)*cos(n*Pi*x/A),x=0..A) ); # the value
of a_n
```

$$a := n \to \mathrm{value}\left(2\,\frac{\int_0^A \mathrm{f}(x)\cos(\frac{n\,\pi\,x}{A})\,dx}{A}\right)$$

The following is the partial Nth sum of the Fourier cosine series

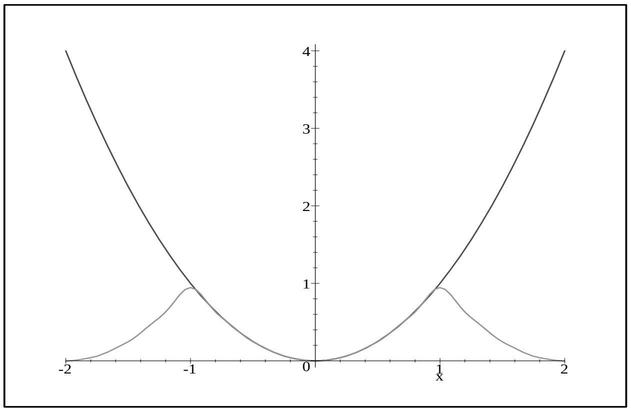
```
>  S:=N->a0+value(Sum(a(n)*cos(n*Pi*x/A),n=1..N) );
```

$$S := N \to a0 + \mathrm{value}\left(\sum_{n=1}^{N} \mathrm{a}(n)\cos(\frac{n\,\pi\,x}{A})\right)$$

```
>  A:=1; #Set a to 1
```

$$A := 1$$

```
>  plot({f(x), S(7)},x=-2..2); #plot of S(7) - the first seven terms
of the Fourier expansion.  x=-1..1
```



3. The following Maple code will generate the sine expansion of $f(x)$ on $-A \le x \le A$. Now keep in mind that the sine expansion will converge to the ODD EXTENSION of $f(x) = x^2$, which of course is NOT the same as x^2 (since x^2 is even). This will be clear from the plot below.

```
>  f:=x->x^2;
```

$$f := x \to x^2$$

```
>  b:=n->value( (2/A)*Int(f(x)*sin(n*Pi*x/A),x=0..A) ); #the value
of b_n
```

$$b := n \to \mathrm{value}\left(2\,\frac{\int_0^A \mathrm{f}(x)\sin(\frac{n\,\pi\,x}{A})\,dx}{A}\right)$$

The following is the partial Nth sum of the Fourier sine series

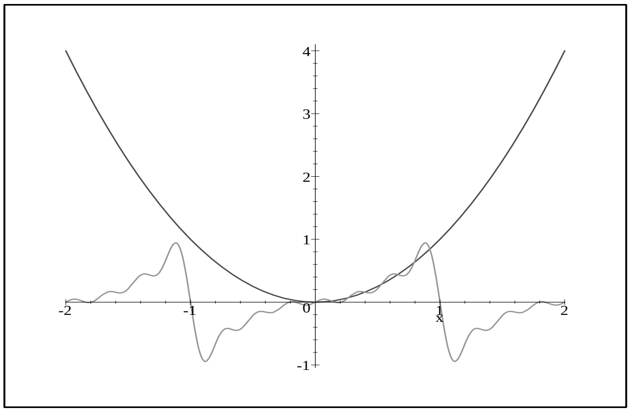
```
> S:=N->value(Sum(b(n)*sin(n*Pi*x/A),n=1..N));
```

$$S := N \to \mathrm{value}\left(\sum_{n=1}^{N} \mathrm{b}(n)\sin(\frac{n\,\pi\,x}{A})\right)$$

```
> A:=1;
```

$$A := 1$$

```
> plot({f(x),S(7)},x=-2..2);
```



Notice that the sine series approximates the odd extension of x^2 and not x^2 itself on $-1 \leq x \leq 0$. Of course outside the interval $-1 \leq x \leq 1$ the sine series approximates the periodic extension of the odd extension of x^2. Also notice how much poorer a sine series approximates the even function f(x)=xˆ2
than does a cosine series (the previous figure).

4. Now for the absolute value of the sine function on $-\pi \leq x \leq \pi$, which we treat as the even extension of $sin(x)$ on the interval $0 \leq x \leq \pi$.

```
> f:=x->abs(sin(x));
```

$$f := x \to |\sin(x)|$$

```
> (1/(2*Pi))*Int(f(x),x=-Pi..Pi); a0:=value("); #the coefficient
of a_0
```

$$\frac{1}{2}\,\frac{\int_{-\pi}^{\pi} |\sin(x)|\,dx}{\pi}$$

$$a0 := \frac{2}{\pi}$$

```
>  a:=n->value( (1/Pi)*Int(f(x)*cos(n*x),x=-Pi..Pi) ); # the value
of a_n
```

$$a := n \to \mathrm{value}\left(\frac{\displaystyle\int_{-\pi}^{\pi} \mathrm{f}(x)\cos(n\,x)\,dx}{\pi}\right)$$

```
>  b:=n->value( (1/Pi)*Int(f(x)*sin(n*x),x=-Pi..Pi) ); #the value
of b_n
```

$$b := n \to \mathrm{value}\left(\frac{\displaystyle\int_{-\pi}^{\pi} \mathrm{f}(x)\sin(n\,x)\,dx}{\pi}\right)$$

The following is the partial Nth sum of the Fourier series

$$0$$

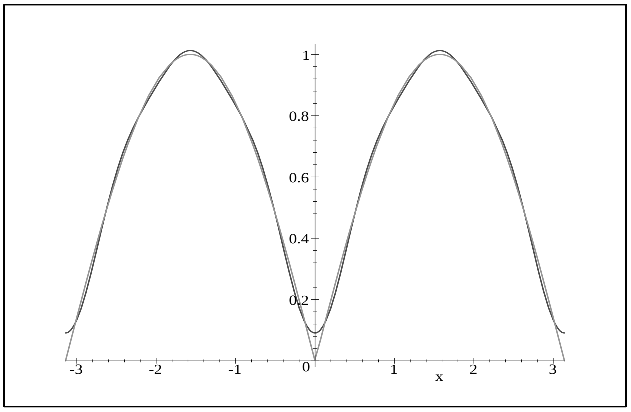
```
>  S:=N->a0+a(1)*cos(x)+value(Sum(a(n)*cos(n*x)+ b(n)*sin(n*x),n=2..N));
```

$$S := N \to a0 + \mathrm{a}(1)\cos(x) + \mathrm{value}(\sum_{n=2}^{N} (\mathrm{a}(n)\cos(n\,x) + \mathrm{b}(n)\sin(n\,x)))$$

```
>  S(4);
```

$$\frac{2}{\pi} - \frac{4}{3}\,\frac{\cos(2\,x)}{\pi} - \frac{4}{15}\,\frac{\cos(4\,x)}{\pi}$$

```
>  plot({f(x), S(7)},x=-Pi..Pi); #plot of S(7) x=-Pi..Pi
```



5. Now for the step function which is 1 on the interval -1/2 <= x <= 1/2 and zero everywhere else. Again, we treat this as the even extension of the function which is 1 on $0 \leq x < 1/2$ and zero everywhere else.

```
> f:=x->x^2; #Now for the step function on 0..1
```

$$f := x \to x^2$$

```
> f:=x->piecewise(x<-1/2,0,x<1/2,1,0);
```

$$f := x \to \text{piecewise}(x < \frac{-1}{2},\, 0,\, x < \frac{1}{2},\, 1,\, 0)$$

```
> Int(1,x=0..1/2); a0:=value(");
```

$$\int_0^{1/2} 1\,dx$$

$$a0 := \frac{1}{2}$$

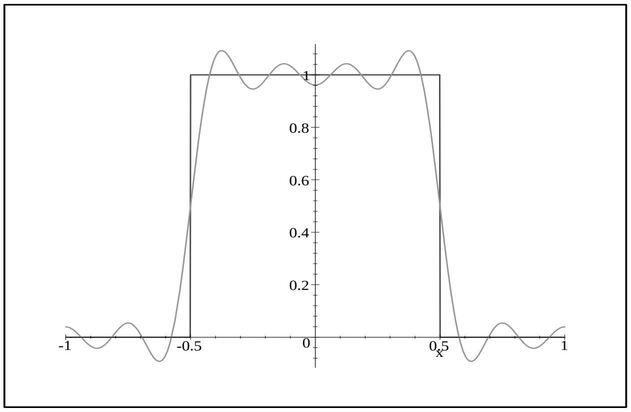
```
> a:=n->value( (2)*Int(1*cos(n*Pi*x),x=0..1/2) );
```

$$a := n \to \text{value}(2\int_0^{1/2} \cos(n\,\pi\,x)\,dx)$$

```
> S:=N->a0+value(Sum(a(n)*cos(n*Pi*x),n=1..N));
```

$$S := N \to a0 + \text{value}(\sum_{n=1}^{N} \mathrm{a}(n)\cos(n\,\pi\,x))$$

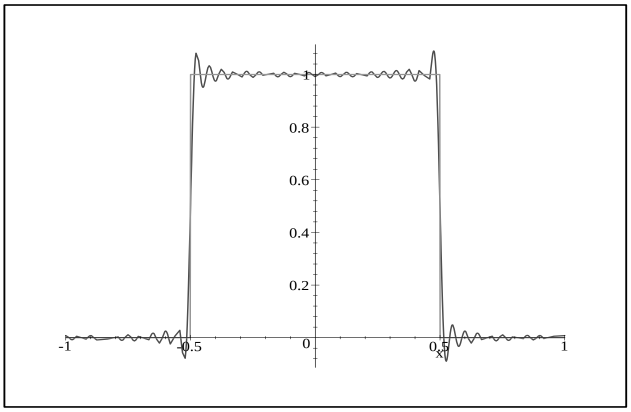
```
> plot({S(7),f(x)}, x=-1..1);
```



Notice how much poorly 7 terms of the Fourier series approximates this discontinuous function than for a continuous function.
Here is the graph of the first 40 terms.

$$\frac{1}{2}$$

```
>  plot({S(40),f(x)}, x=-1..1);
```



6. Now f(x) = exp(rx) on the interval $-\pi \le x \le \pi$.

```
>  f:=x->exp(r*x); A:=Pi; r:=1/2;
```

$$f := x \to e^{(r\,x)}$$

$$A := \pi$$

$$r := \frac{1}{2}$$

```
>  (1/(2*A))*Int(f(x),x=-A..A); a0:=value("); #the coefficient of
a_0
```

$$\frac{1}{2}\,\frac{\int_{-\pi}^{\pi} e^{(1/2\,x)}\,dx}{\pi}$$

$$a0 := \frac{1}{2}\,\frac{2\,e^{(1/2\,\pi)} - 2\,e^{(-1/2\,\pi)}}{\pi}$$

```
>  a:=n->value( (1/A)*Int(f(x)*cos(n*Pi*x/A),x=-A..A) ); # the value
of a_n
```

$$a := n \to \mathrm{value}\left(\frac{\int_{-A}^{A} \mathrm{f}(x)\cos(\frac{n\,\pi\,x}{A})\,dx}{A}\right)$$

```
>  b:=n->value( (1/A)*Int(f(x)*sin(n*Pi*x/A),x=-A..A) ); #the value
of b_n
```

$$b := n \to \mathrm{value}\left(\frac{\int_{-A}^{A} \mathrm{f}(x)\sin(\frac{n\,\pi\,x}{A})\,dx}{A}\right)$$

The following is the partial Nth sum of the Fourier series

```
>  S:=N->a0+value(Sum(a(n)*cos(n*Pi*x/A)+b(n)*si n(n*Pi*x/A),n=1..N));
```

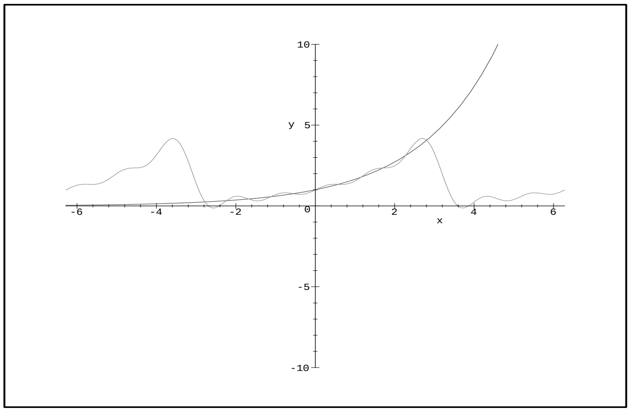
$$S := N \to a0 + \mathrm{value}\left(\sum_{n=1}^{N} (\mathrm{a}(n)\cos(\frac{n\,\pi\,x}{A}) + \mathrm{b}(n)\sin(\frac{n\,\pi\,x}{A}))\right)$$

$$r = \frac{1}{2}$$

```
>  S(3):
```

Here is a plot of the first 5 terms of its Fourier series over the interval from -2Pi to 2Pi.

```
>  plot({f(x),S(5)},x=-2*Pi..2*Pi,y=-10..10);
```



8. Let $f(x) = \cos(x)$. Expand this in a sine series on the interval $0 \leq x \leq \pi$.

```
>  f:=x->cos(x); A:=Pi;
```

$$f := \cos$$

$$A := \pi$$

```
>  b:=n->value( (2/A)*Int(f(x)*sin(n*Pi*x/A),x=0..A) ); #the value
of b_n
```

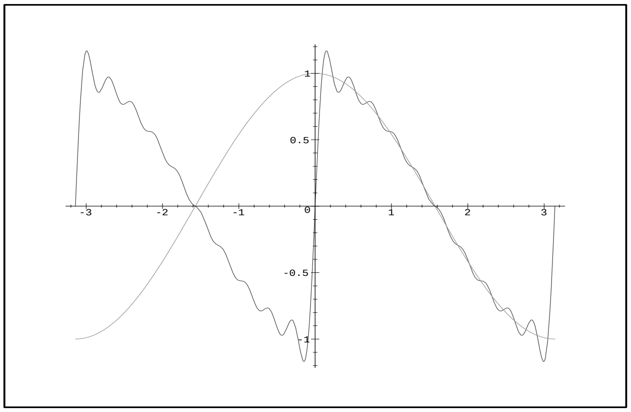
$$b := n \to \mathrm{value}\left(2\,\frac{\int_0^A \mathrm{f}(x)\sin(\frac{n\,\pi\,x}{A})\,dx}{A} \right)$$

The following is the partial Nth sum of the Fourier sine series of cosine(x). The formula for b(n) is only accurate for n > 1 (b(1) is easily shown to be zero). So the sum below starts at n=2.

```
>  S:=N->value(Sum(b(n)*sin(n*Pi*x/A),n=2..N));
```

$$S := N \to \mathrm{value}\left(\sum_{n=2}^{N} \mathrm{b}(n)\sin(\frac{n\,\pi\,x}{A}) \right)$$

```
>  plot({f(x),S(20)},x=-Pi..Pi,numpoints=100);
```



The sine expansion approximates cosine on $0 \le x \le \pi$.
It approximates the odd extension of cosine on the interval
$-\pi \le x \le \pi$.

10. Let $f(x) = \sin(n * x)/\sqrt{\pi}$ and $g(x) = \sin(m * x)/\sqrt{\pi}$. As shown in class

$$\int_{-\pi}^{\pi} f(x)g(x)\,dx = \begin{array}{ll} 0 & \text{if } n \ne m \\ 1 & \text{if } n = m \end{array}$$

We let $t = ax/\pi$ in the integral on the right. As x ranges from $-\pi$ to π, t ranges from $-a$ to a. Also $dx = \pi dt/a$. So this change of variables transforms the integral on the left to

$$\int_{-a}^{a} f(\pi t/a)g(\pi t/a)\frac{\pi dt}{a} = \int_{-a}^{a} \sqrt{\frac{\pi}{a}} f(\pi t/a)\sqrt{\frac{\pi}{a}} g(\pi t/a)\,dt.$$

Thus the set of functions, $\sqrt{\frac{\pi}{a}}f(\pi t/a) = \frac{1}{\sqrt{a}}\sin(n\pi t/a)$, (as n ranges from 0 to ∞) is orthonormal. Adding the cosines in follows the same pattern.

11. Lemma 3 was proved in class. As for Theorem 12, we first show orthonormality. Let $f_n(t) = \frac{1}{\sqrt{2a}}e^{in\pi t/a}$ for $n \in Z$. If $n \neq m$, we have

$$\begin{aligned} \langle f_n, f_m\rangle &= \frac{1}{2a}\int_{-a}^{a} e^{i(n-m)\pi t/a}\,dt \\ &= \frac{a}{2a\pi i(n-m)}e^{i(n-m)\pi t/a}|_{-a}^{a} \\ &= 0 \end{aligned}$$

where the last equality holds because $e^{i(n-m)\pi t/a}$ is periodic with period $2a$. If $n = m$, then the above integral is

$$\frac{1}{2a}\int_{-a}^{a} 1\,dt$$

which evaluates to 1, as desired.

As for spanning, suppose f is a continous, piecewise differentiable function on the interval $-a \le x \le a$. Letting $t = \pi x/a$, or equivalently $x = ta/\pi$, we obtain a function $f(ta/\pi)$, which is defined on the interval $-\pi \le t \le \pi$. We showed in class that the functions $\{e^{int},\ n \in Z\}$ span the functions on the interval $-\pi \le t \le \pi$. Therefore $f(ta/\pi)$ is spanned by the functions $\{e^{int},\ n \in Z\}$. Now transfer back to the interval $-a \le x \le a$ by $t = \pi x/a$, or equivalently $x = ta/\pi$, and we see that $f(x)$ is spanned by $\{e^{in\pi x/a},\ n \in Z\}$, as desired.

17. The periodic extension of the function in part a) is continuous everywhere and is piecewise differentiable. So its Fourier series converges uniformly (and so also pointwise and in L^2).

The periodic extension of the function in part b) is NOT continuous at integer values. So its Fourier series converges pointwise at non integer values to the periodic extension of f. At integer values, the Fourier series equals zero (the average of the left and right limits of f there). The Fourier series does converge uniformly on intervals of the form $-\lambda \le x \le \lambda$ where λ is a fixed number less than one. Intervals of this type avoid the discontinuities of f. The Fourier series does converge to f in L^2.

Part c) is similar to b). The Fourier series converges pointwise to f away from the discontinuities. At a discontinuity, such as $x = 1/2$ or $x = -1/2$, the Fourier series converges to $1/2$ which is the average of the left and right limits. The Fourier series does converge to f in L^2.

18. The discussion on page 59 (right after (2.16) shows that $a_n = -b'_n/n$ where b'_n is the Fourier sine coefficient of f'. Likewise, $b_n = a'_n/n$ where a'_n is the Fourier cosine coefficient of f'. Therefore,

$$\sum_{n=0}^{\infty} |a_n|^2 + |b_n|^2 = |a_0|^2 + \sum_{n=1}^{\infty} \frac{|a'_n|^2}{n^2} + \frac{|b'_n|^2}{n^2}$$

Since f' is continuous (and therefore bounded) the $|a_n'|$ and $|b_n'|$ are bounded above by some number, M. Thus,

$$\sum_{n=0}^{\infty} |a_n|^2 + |b_n|^2 \leq |a_0|^2 + M \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{n^2} < \infty$$

Now let f_N be the Fourier partial series up through order N. Then

$$\begin{aligned} ||f - f_N||_{L^2}^2 &= \langle \sum_{n=N+1}^{\infty} a_n \cos(nx) + b_n \sin(nx), \sum_{m=N+1}^{\infty} a_m \cos(mx) + b_m \sin(mx) \rangle_{L^2} \\ &= \sum_{n=N+1}^{\infty} |a_n|^2 \langle \cos nx, \cos nx \rangle + |b_n|^2 \langle \sin nx, \sin nx \rangle \end{aligned}$$

where the last equality follows since $\cos nx$ and $\sin mx$ are mutually orthogonal. By the computation in class, the latter two inner products are both π. Therefore

$$||f - f_N||^2 = \pi \sum_{n=N+1}^{\infty} |a_n|^2 + |b_n|^2$$

Since the series $\sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2$ converges to a finite number as shown above, the tail of this series (from $N+1$ to ∞) gets small (as $N \mapsto \infty$). Therefore, $||f - f_N||_{L^2} \mapsto 0$ as $N \mapsto \infty$, and so f_N converges to f in the mean as desired.