Solutions to Chapter 3

1. Summing the two trig identities that are given in the problem yields

$$\cos mt \cos \lambda t = \left(\cos(\lambda + m)t + \cos((\lambda - m)t\right)/2$$

Integrating the expression on the right gives

$$\int_{-\pi}^{\pi} \cos mt \cos \lambda t \, dt = \frac{1}{2}\left(\frac{\sin(\lambda+m)t)}{\lambda+m} + \frac{\sin(\lambda-m)t}{\lambda-m}\right)|_{-\pi}^{\pi}$$

Inserting the limits and simplifying yields

$$\frac{-2(-1)^m \lambda \sin(\lambda\pi)}{(m^2-\lambda^2)}$$

Setting $m = 3$ shows that the Fourier transform of $\cos(3t)$ (from $-\pi$ to π) is

$$-\frac{\sqrt{2}\,\lambda \sin(\pi\,\lambda)}{\sqrt{\pi}\,(-9+\lambda^2)}$$

as claimed.

2. Since sin is an odd function, we only need to compute the sine part of the Fourier transform. That is, we must compute

$$\frac{-i}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin 3t \sin \lambda t \, dt$$

Subtracting the two trig identities given in problem 1, gives

$$\frac{-i}{\sqrt{2\pi}} \sin 3t \sin \lambda t = \frac{-i}{\sqrt{2\pi}} \left(\cos(\lambda-3)t) - \cos(\lambda+3)t\right)/2$$

which when integrated from $-\pi$ to π gives

$$\frac{i}{2\sqrt{2\pi}}\left(\frac{\sin(\lambda-3)t}{\lambda-3} - \frac{\sin(\lambda+3)t}{\lambda+3}\right)|_{-\pi}^{\pi} = 3\,\frac{i\,\sqrt{2}\sin(\pi\,\lambda)}{\sqrt{\pi}\,(-3+\lambda)\,(3+\lambda)}$$

3. The Maple code for displaying f and computing its Fourier transform is given below.

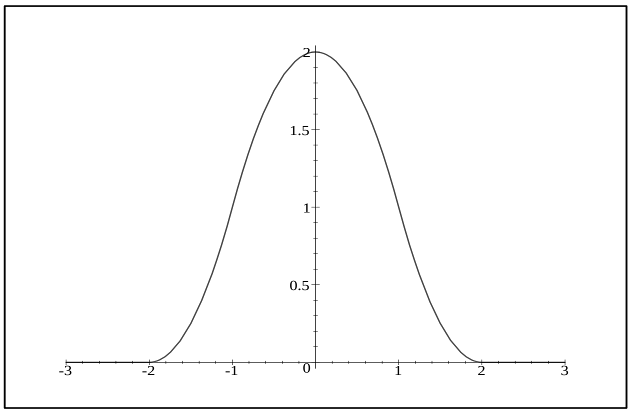
```
>   f:=t-> piecewise(t<-2,0,t<-1,t^2+4*t+4,t<1,2-t^2,t<2,t^2-4*t+4,0);
```

$f := t \to \mathrm{piecewise}(t < -2,\, 0,\, t < -1,\, t^2{+}4\,t{+}4,\, t < 1,\, 2{-}t^2,\, t < 2,\, t^2{-}4\,t{+}4,\, 0)$

```
> f(t);
```

$$\begin{cases} 0 & t < -2 \\ t^2 + 4\,t + 4 & t < -1 \\ 2 - t^2 & t < 1 \\ t^2 - 4\,t + 4 & t < 2 \\ 0 & \textit{otherwise} \end{cases}$$

```
> plot(f,-3..3);
```



Note that f is continous. Each piece matches up at the boundary points. For example the limit of $t^2 + 4t + 4$ as t approaches -2 is zero, which is the same as the value of f on the interval $t < -2$. The other pieces match up similarly.

The derivative is given by

```
> D(f)(t);
```

$$\begin{cases} 0 & t < -2 \\ 2\,t + 4 & t < -1 \\ -2\,t & t < 1 \\ 2\,t - 4 & t < 2 \\ 0 & \textit{otherwise} \end{cases}$$

The derivative is also continuous since each of its pieces match up at the boundary defining points. For example, the limit of $2t+4$ as t approaches -2 is 0 which agrees with f' for $t < -2$.

Here is the computation of the Fourier transform. Since f is even, we need only compute the cosine part of the transform.

```
> (1/sqrt(2*Pi))*Int(f(t)*cos(lambda*t),t=-2..2 ); value(");
```

$$\frac{1}{2}\,\frac{\sqrt{2}\int_{-2}^{2}\left(\begin{cases}0 & t<-2\\ t^2+4\,t+4 & t<-1\\ 2-t^2 & t<1\\ t^2-4\,t+4 & t<2\\ 0 & \mathit{otherwise}\end{cases}\right)\cos(\lambda\,t)\,dt}{\sqrt{\pi}}$$

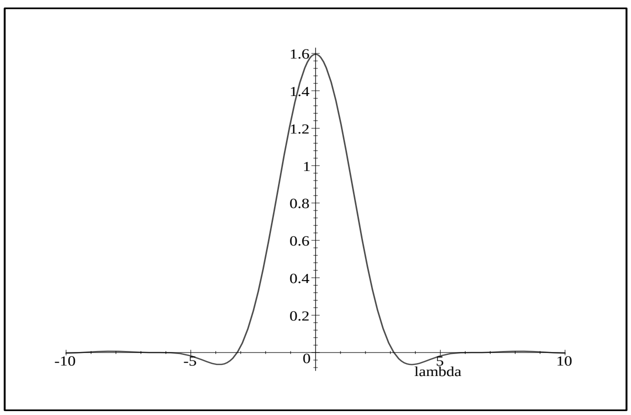
$$-4\,\frac{\sqrt{2}\sin(\lambda)\,(\cos(\lambda)-1)}{\sqrt{\pi}\,\lambda^3}$$

```
> trans:=";
```

$$\mathit{trans} := -4\,\frac{\sqrt{2}\sin(\lambda)\,(\cos(\lambda)-1)}{\sqrt{\pi}\,\lambda^3}$$

Here is the plot of the transform over the interval $-10 \le \lambda \le 10$.

```
> plot(trans,lambda=-10..10);
```



Note that the Fourier transform decays like $1/\lambda^3$. There is a pattern here. A function which has finite support and bounded leads to a Fourier transform that decays like $1/\lambda$ (see example 21 in the text, which was also done in class). If the function is also continous then its transform decays like $1/\lambda^2$ (such as in problem 1). If the function also has a continous derivative, then the transform decays like $1/\lambda^3$ as in this exercise.

5. To prove the $\mathcal{F}^{-1}$ is linear, we have

$$\begin{aligned}\mathcal{F}^{-1}[f+g](x) &= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}[f(\lambda)+g(\lambda)]e^{i\lambda x}\,d\lambda\\ &= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}[f(\lambda)]e^{i\lambda x}\,d\lambda + \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}[g(\lambda)]e^{i\lambda x}\,d\lambda\\ &= \mathcal{F}^{-1}[f](x)+\mathcal{F}^{-1}[g](x)\end{aligned}$$

To prove

$$\mathcal{F}^{-1}[\lambda^n f(\lambda)](t) = (-i)^n \frac{d^n}{dt^n}\{\mathcal{F}^{-1}[f](t)\}$$

We start with the right side:

$$\begin{aligned}
(-i)^n \frac{d^n}{dt^n}\{\mathcal{F}^{-1}[f](t)\} &= (-i)^n \frac{d^n}{dt^n}\left\{\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}[f(\lambda)]e^{i\lambda t}\,d\lambda\right\} \\
&= (-i)^n \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}[f(\lambda)]\frac{d^n}{dt^n}\left\{e^{i\lambda t}\right\}\,d\lambda \\
&= (-i)^n \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}[f(\lambda)](i\lambda)^n e^{i\lambda t}\,d\lambda \\
&= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty}[f(\lambda)](\lambda)^n e^{i\lambda t}\,d\lambda \\
&= \mathcal{F}^{-1}[\lambda^n f(\lambda)](t)
\end{aligned}$$

To prove

$$\mathcal{F}^{-1}[f^{(n)}(\lambda)](t) = (-it)^n \mathcal{F}^{-1}[f](t)$$

we start with the left side with $n = 1$:

$$\mathcal{F}^{-1}[f'(\lambda)](t) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} f'(\lambda)e^{i\lambda t}\,d\lambda$$

We integrate by parts with $dv = f'$ and $u = e^{i\lambda t}$ to obtain

$$\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} f'(\lambda)e^{i\lambda t}\,d\lambda = \frac{1}{\sqrt{2\pi}}\left(f(\lambda)e^{i\lambda t}|_{\lambda=-\infty}^{\infty} - \int_{\infty}^{\infty} f(\lambda)(ite^{i\lambda t})\,d\lambda\right)$$

The boundary terms are zero since $f(\lambda) = 0$ for large λ. Therefore

$$\mathcal{F}^{-1}[f'(\lambda)](t) = -\int_{\infty}^{\infty} f(\lambda)(ite^{i\lambda t})\,d\lambda = -it\mathcal{F}^{-1}[f(\lambda)](t)$$

To show the statement regarding the Laplace transform, we have

$$\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(t)e^{-i\lambda t}\,dt$$

Since $f(t) = 0$ for $t < 0$, the lower limit of integration starts at 0 rather than $-\infty$. On the other hand

$$\frac{1}{\sqrt{2\pi}}\mathcal{L}f(i\lambda) = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(t)e^{-ti\lambda}\,dt$$

which is the same as $\mathcal{F}[f](\lambda)$.

8. Here is the proof of Theorem 3.12

Shifts. We have

$$\begin{aligned}\widehat{z}_j &= \sum_{k=0}^{n-1} z_k \overline{w}^{kj} \quad \text{where } w = e^{i2\pi/n} \\ &= \sum_{k=0}^{n-1} y_{k+1} \overline{w}^{kj}\end{aligned}$$

We change indices by letting $\ell = k+1$ to obtain

$$\begin{aligned}\widehat{z}_j &= \sum_{\ell=1}^{n} y_l \overline{w}^{(l-1)j} \\ &= \overline{w}^{-j} \sum_{\ell=1}^{n} y_l \overline{w}^{lj} \\ &= w^j \sum_{\ell=1}^{n} y_l \overline{w}^{lj}\end{aligned}$$

where the last equality holds since $\overline{w}^{-1} = w$ (recall that $w = e^{2\pi i/n}$). The last line is $w^j \widehat{y}_j$ as desired.

Convolutions. We must show

$$(x * y)_{k+n} = (x * y)_k$$

The left side is

$$(x * y)_{k+n} = \sum_{j=0}^{n-1} x_j y_{k+n-j}$$

Since $y_{k+n-j} = y_{k-j}$, we have

$$(x * y)_{k+n} = \sum_{j=0}^{n-1} x_j y_{k+n-j} = \sum_{j=0}^{n-1} x_j y_{k-j} = (x * y)_k$$

Convolution Theorem. We have

$$\begin{aligned}(\widehat{x * y})_\ell &= \sum_{k=0}^{n-1} (x * y)_k \overline{w}^{\ell k} \\ &= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} x_{k-j} y_j \overline{w}^{\ell k} \\ &= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} x_{k-j} \overline{w}^{\ell(k-j)} y_j \overline{w}^{\ell j} \\ &= \sum_{j=0}^{n-1} \sum_{t=-j}^{n-1-j} x_t \overline{w}^{t\ell} y_j \overline{w}^{\ell j} \quad \text{with } t = k-j \\ &= \sum_{j=0}^{n-1} \sum_{t=0}^{n-1} x_t \overline{w}^{t\ell} y_j \overline{w}^{\ell j}\end{aligned}$$

where the last equation follows from the fact that $x_{q-n} = x_q$ and $w^{(q-n)\ell} = w^{q\ell}$ (so the inner sum from $t = -j$ to $t = -1$ is the same as the sum from $t = n - j$ to $n - 1$). The last expression on the right is

$$\sum_{t=0}^{n-1} x_t \overline{w}^{t\ell} \sum_{j=0}^{n-1} y_j \overline{w}^{\ell j} = \widehat{x}_\ell \widehat{y}_\ell$$

as desired.

Last Property. We have

$$\widehat{y}_{n-k} = \sum_{j=0}^{n-1} y_j \overline{w}^{j(n-k)}$$

Now use the fact that w^n=1 and that $\overline{w}^{-jk} = w^{jk}$ (again, recall that $w = e^{2\pi i/n}$). We obtain

$$\widehat{y}_{n-k} = \sum_{j=0}^{n-1} y_j w^{jk}$$

Since the y_j are real, the right side is

$$\overline{\sum_{j=0}^{n-1} y_j \overline{w}^{jk}} = \overline{\widehat{y}_k}$$

as desired.