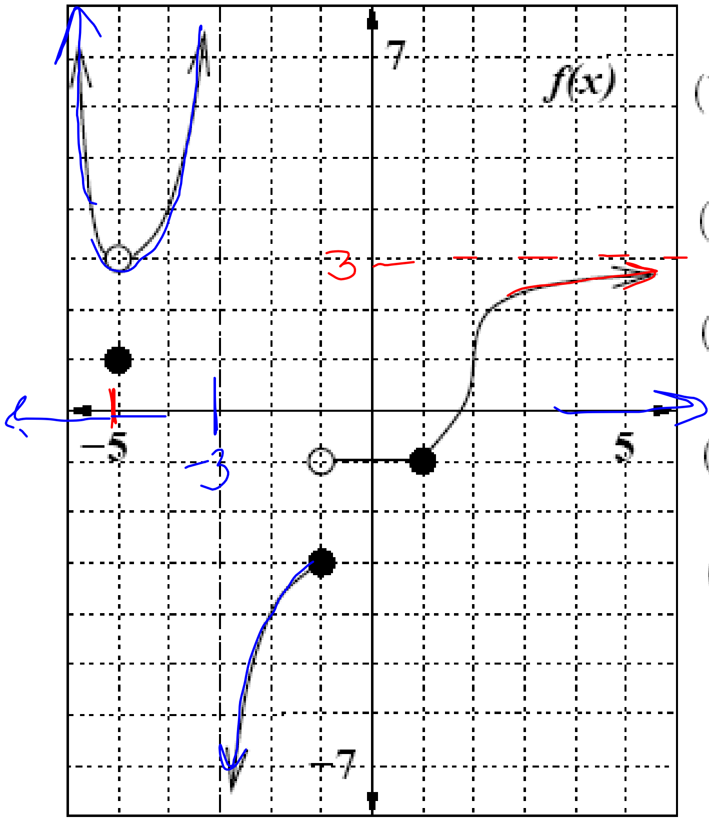
Week-In-Review #5

1. Use the given graph of $f(x)$ to answer the following questions.



from the left

(a) $\lim_{x \to -1^-} f(x) = -3$

(b) $\lim_{x \to -1^+} f(x) = -1$

$\neq$

from the right

(c) $\lim_{x \to -1} f(x) =$ DNE

from both sides

(d) $\lim_{x \to -3^+} f(x) = -\infty$

(e) $\lim_{x \to 0} f(x) = -1$

(f) $\lim_{x \to 1^+} f(x) = -1$

(g) $\lim_{x \to -5} f(x) = 3$

$f(-5) = 1$

(h) $\lim_{x \to \infty} f(x) = 3$

(i) $\lim_{x \to -\infty} f(x) = +\infty$

Find the following limits:

2. $\lim_{x\to 1} \boxed{\frac{2x^2+5x-4}{x+2}} = \frac{2(1)^2+5(1)-4}{1+2} = \frac{2+5-4}{3} = \frac{3}{3} = \boxed{1}$

3. $\lim_{x\to 5} \frac{x^2-3x-10}{x-5} = \frac{5^2-3(5)-10}{5-5} = \frac{25-15-10}{0} = \frac{0}{0}$

$\lim_{x\to 5} \frac{(x+2)\cancel{(x-5)}}{\cancel{x-5}} = \lim_{x\to 5} x+2 = 5+2 = \boxed{7}$

4. $\lim_{h\to 0} \frac{2x+3h}{x} = \frac{2x+3(0)}{x} = \frac{2x}{x} = \boxed{2}$

5. $\lim_{x\to 0} \frac{x^4}{x^3+2} = \frac{0^4}{0^3+2} = \frac{0}{2} = \boxed{0}$

6. $\lim_{x\to 3} \frac{|x-3|}{x-3} = \frac{0}{0}$

$\downarrow$ $\boxed{\text{DNE}}$

NUMERICALLY

From the left $(x = 2.9999)$

$\frac{|2.9999-3|}{2.9999-3} = \frac{+0.0001}{-0.0001} = -1$

From the right $(x = 3.0001)$

$\frac{|3.0001-3|}{3.0001-3} = \frac{+0.0001}{+0.0001} = 1$

$\neq$

7. Given $f(x) = \begin{cases} x+3 & , x < -1 \\ 0 & , x = -1 \\ x^2+1 & , x > -1 \end{cases}$ find

x	$x+3$
-1	2
-2	1
-3	0

x	x^2+1
-1	2
0	1
1	2
2	5

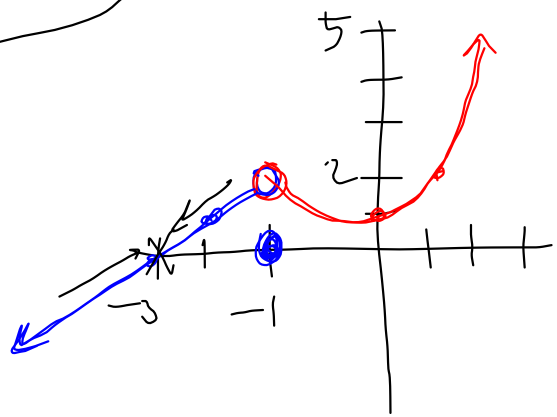
(a) $\lim_{x\to -1^-} f(x) = \lim_{x\to -1} x+3 = -1+3 = \boxed{2}$

(b) $\lim_{x\to -1^+} f(x) = \lim_{x\to -1} x^2+1 = (-1)^2+1 = \boxed{2}$

(c) $\lim_{x\to -1} f(x) = \boxed{2}$

Func. value

$f(-1) = 0$



(d) $\lim_{x\to -3} f(x) = \lim_{x\to -3} x+3 = -3+3 = \boxed{0}$

(e) $\lim_{x\to 5} f(x) = \lim_{x\to 5} x^2+1 = 5^2+1 = \boxed{26}$

8. $\lim_{x\to -1^+} \frac{x+2}{x^2+3x+2} = \frac{-1+2}{(-1)^2+3(-1)+2} = \frac{1}{0}$

$\lim_{x\to -1^+} \frac{\cancel{x+2}}{(\cancel{x+2})(x+1)} = \lim_{x\to -1^+} \frac{1}{x+1}$

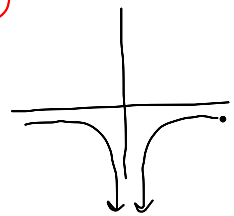
-1 $\quad x=-.9999$

$\frac{-.9999+2}{(-.9999)^2+3(-.9999)+2}$

$\frac{1}{\underbrace{-.9999+1}} = \frac{1}{.0001} = \frac{1}{0^+} = \boxed{+\infty}$

9. $\lim_{x\to 0^-} \frac{-5}{x^2} = \frac{-5}{(-0.0001)(-0.0001)} = \frac{-5}{0^+} = \boxed{-\infty}$

$x=-0.0001$



10. $\lim_{x\to\infty} \frac{2x^2+4x-x^3}{3+7x^3+8x} = \boxed{-\frac{1}{7}}$

HA?

$\lim_{x\to\pm\infty} f(x) = L$

$\Rightarrow$ HA: $y=L$

end behavior

11. $\lim_{x\to-\infty} \frac{x^2+2x+5}{2x^2+3x^4} = \boxed{0}$

HA?

12. $\lim_{x\to-\infty} \frac{x^2+2x}{\boxed{x}+5} = \frac{(-\infty)^2+2(-\infty)}{-\infty+5} = \frac{+\infty-\infty}{-\infty+5} = \frac{\infty}{-\infty}$

HA?

NO HA

$\lim_{x\to-\infty} \frac{(x^2+2x)(\frac{1}{x})}{(x+5)(\frac{1}{x})} = \lim_{x\to-\infty} \frac{x+2}{1+\frac{5}{x}} = \frac{-\infty+2}{1+0} = \frac{-\infty}{1} = \boxed{-\infty}$

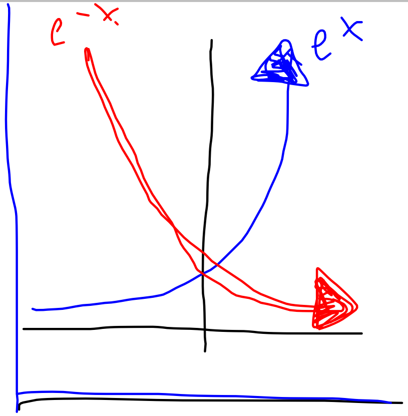
$\frac{\infty}{2} \to \infty$

$\frac{2}{\infty} \to 0$

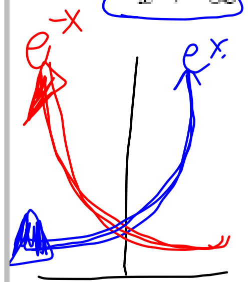
13. $\lim_{x\to\infty} \frac{2e^x+3e^{-x}}{5e^{-x}-7e^x} = \frac{2(\infty)+3(0)}{5(0)-7(\infty)} = \frac{\infty}{-\infty}$

HA?

$$\lim_{x\to\infty}\frac{\frac{2e^x}{e^x}+\frac{3e^{-x}}{e^x}}{\frac{5e^{-x}}{e^x}-\frac{7e^x}{e^x}} = \lim_{x\to\infty}\frac{2+3e^{-2x}}{5e^{-2x}-7} = \frac{2+0}{0-7} = \boxed{-\frac{2}{7}}$$



14. $\lim_{x\to-\infty} \frac{2e^x+3e^{-x}}{5e^{-x}-7e^x} = \frac{2(0)+3(\infty)}{5(\infty)-7(0)} = \frac{\infty}{\infty}$



$$\lim_{x\to-\infty}\frac{\frac{2e^x}{e^{-x}}+\frac{3e^{-x}}{e^{-x}}}{\frac{5e^{-x}}{e^{-x}}-\frac{7e^x}{e^{-x}}} = \lim_{x\to-\infty}\frac{2e^{2x}+3}{5-7e^{2x}} = \boxed{\frac{3}{5}}$$

$$\frac{e^{-x}}{e^x} = e^{-x}\cdot e^{-x} = e^{-2x}$$

$$\frac{e^x}{e^{-x}} = e^x\cdot e^x = e^{2x}$$

$$\lim_{x\to\infty}\frac{2}{e^x+3e^{-x}} = \frac{2}{\infty+3(0)} = \frac{2}{\infty} = \boxed{0}$$

15. Given $f(x) = \begin{cases} \frac{2x^2+1}{x^2+1}, & x<0 \\ \frac{x+5}{x^3+8}, & x\geq 0 \end{cases}$, find

(a) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{x+5}{x^3+8} = \boxed{0}$

(b) $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{2x^2+1}{1x^2+1} = \boxed{2}$

16. Where are the following functions continuous?

(a) $f(x) = 3x^7 + 8x^5 - 8x + 10$ polynomial

$\boxed{(-\infty, \infty)}$

(b) $f(x) = 5e^{(x-2)} + 17$ exponential

$\boxed{(-\infty, \infty)}$

(c) $f(x) = \dfrac{x^2-4}{x^3-8}$ Rational → continuous on its domain

$x^3 - 8 \neq 0$

$x^3 \neq 8$

$x \neq 2$

$\boxed{(-\infty, 2) \cup (2, \infty)}$

(d) $f(x) = 2\ln\boxed{2x-7} + 5$ logarithmic → cont. on its domain

$2x - 7 > 0$

$2x > 7$

$x > 7/2$

$\boxed{\left(\frac{7}{2}, \infty\right)}$

e) $f(x) = \sqrt{\boxed{x+8}}$

$x + 8 \geq 0 \quad x \geq -8$

$\boxed{[-8, \infty)}$

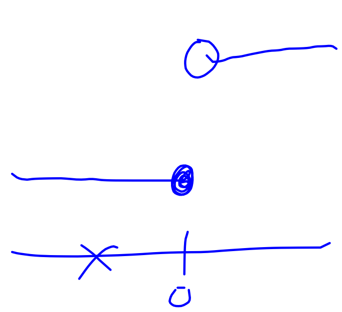
$f(x)$ Continuous at $x = a$:

1. Function value at $x = a$: $f(a)$ exists
2. Limit value at $x = a$: $\lim_{x \to a} f(x)$ exists
3. Limit value = function value at $x = a$: $\lim_{x \to a} f(x) = f(a)$

17. Where is the following function discontinuous?

$\boxed{\text{Not cont @ } x=-3 \text{ \& } x=0}$

$$f(x) = \begin{cases} \dfrac{x+1}{x^2-9}, & x \le 0 \\ 2x+4, & x > 0 \end{cases}$$



$\boxed{x>0}$: $2x+4$

① Check each piece for discontinuities:

$\boxed{x \le 0}$: $\dfrac{x+1}{x^2-9}$ $x^2-9 \ne 0$
$x^2 \ne 9$
$x \ne \pm 3$ $\Rightarrow x \ne -3$

$\boxed{\text{Not cont. @ } x=-3}$

② Check that pieces "meet" at a function value:
We are looking at & around $x=0$ in this example

Function value : $f(0) = \dfrac{0+1}{0^2-9} = \dfrac{1}{-9}$

Limit value : Left : $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0} \dfrac{x+1}{x^2-9} = \dfrac{0+1}{0^2-9} = \dfrac{-1}{9}$

Rt : $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} 2x+4 = 2(0)+4 = 4$

$\ne$ no limit value

$\Rightarrow \boxed{\text{Not cont @ } x=0}$

18. Find the values of a and b so that $g(x)$ is continuous on $(-\infty, \infty)$, if possible.

$$f(x) = \begin{cases} 2a + 6x & , x < 1 \\ 4 & , x = 1 \\ x - b & , x > 1 \end{cases}$$

Check Each piece cont:

$x<1$: $2a+6x$ ✓

$x=1$: 4 ✓

$x>1$: $x-b$ ✓

Need all pieces to meet at func. value when $x=1$:

Function value : $f(1) = 4$

Lim from left : $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1} 2a+6x$

$= 2a+6(1)$

$= 2a+6$

Lim from rt : $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1} x-b$

$= 1-b$

These limits must both be = to 4 (the func. value):

$2a+6=4$ and $1-b=4$

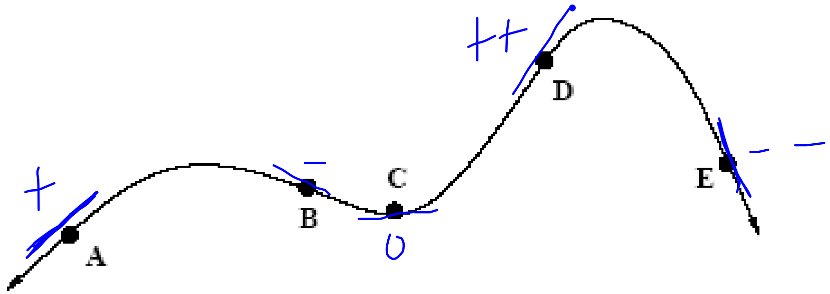
$2a=-2$ $\quad$ $-b=3$

$\boxed{a=-1}$ $\quad$ $\boxed{b=-3}$

19.

(a) Between which consecutive pairs of points in the following graph is the average rate of change positive? negative? zero?

(b) Put the points labeled in the following graph in order (from smallest to largest) according to the value of the instantaneous rate of change of the function at that point.



Avg. Rate of Chg (AROC) = Slope of the secant line thru two pts on a graph.

(+)	(−)	zero
AB	BC	none
CD	DE	

Inst. Rate of Chg (IROC) = slope of the tangent to a particular pt. on a graph

E, B, C, A, D

20. Given $f(x) = x^3 - x + 3$, evaluate the following:

$f(x) = (x)^3 - (x) + 3$

(a) $f(1) = 1^3 - 1 + 3 = \boxed{3}$

(b) $f(a) = \boxed{a^3 - a + 3}$

(c) $f(\star) = \boxed{\star^3 - \star + 3}$

(d) $f(\triangle) = \boxed{\triangle^3 - \triangle + 3}$

(e) $f(Kathryn) = \boxed{K^3 - K + 3}$

(f) $f(Kathryn + Bollinger) = \boxed{(K+B)^3 - (K+B) + 3}$

(g) $f(Math + x) = (math + x)^3 - (math + x) + 3$

(h) $f(x + h) = \boxed{(x+h)^3 - (x+h) + 3}$

$\underbrace{x^3 - x + 3}_{f(x)} + h$
$f(x) + h$

21. Using the limit definition of derivative, find the derivatives of the following functions:

$f'(x) = \text{IROC} = m_{tan}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

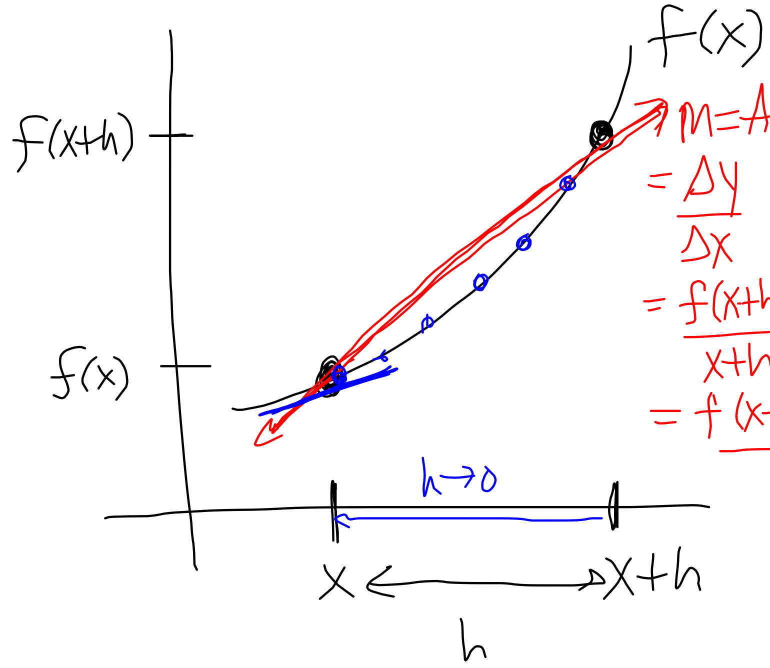
(a) $f(x) = (x)^2 + 3(x) - 7$

$$f(x+h) = (x+h)^2 + 3(x+h) - 7$$
$$= x^2 + 2xh + h^2 + 3x + 3h - 7$$

$f(x+h) - f(x)$

$$= \cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h \cancel{-7} - (\cancel{x^2} + \cancel{3x} \cancel{-7})$$
$$= 2xh + h^2 + 3h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2 + 3h}{h}$$
$$= \frac{\cancel{h}(2x + h + 3)}{\cancel{h}}$$
$$= 2x + h + 3$$



$m = \text{AROC}$
$= \dfrac{\Delta y}{\Delta x}$
$= \dfrac{f(x+h) - f(x)}{x+h-x}$
$= \dfrac{f(x+h) - f(x)}{h}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 2x + h + 3 = 2x + 3$$

$$\boxed{f'(x) = 2x + 3}$$

(b) $g(x) = \frac{(x)}{(x) - 2}$

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\left[\frac{x+h}{x+h-2}\right] - \left[\frac{x}{x-2}\right]}{h}$$

$$= \lim_{h\to 0} \left(\frac{1}{h}\right)\left[\frac{(x+h)(x-2)}{(x+h-2)(x-2)} - \frac{x\,(x+h-2)}{(x-2)(x+h-2)}\right]$$

$$= \lim_{h\to 0} \left(\frac{1}{h}\right)\left[\frac{(x+h)(x-2) - x(x+h-2)}{(x+h-2)(x-2)}\right]$$

$$= \lim_{h\to 0} \left(\frac{1}{h}\right)\left[\frac{x^2-2x+xh-2h-(x^2+xh-2x)}{(x+h-2)(x-2)}\right]$$

$$= \lim_{h\to 0} \left(\frac{1}{h}\right)\left[\frac{\cancel{x^2}\cancel{-2x}\cancel{+xh}-2h\cancel{-x^2}\cancel{-xh}\cancel{+2x}}{(x+h-2)(x-2)}\right] = \lim_{h\to 0}\left(\frac{1}{\cancel{h}}\right)\left[\frac{-2\cancel{h}}{(x+h-2)(x-2)}\right]$$

$$\left(\frac{1}{h}\right)\left(\frac{-2h}{(x+h-2)(x-2)}\right)$$

$$= \frac{(1)(-2\cancel{h})}{(\cancel{h})(x+h-2)(x-2)}$$

$$= \lim_{h\to 0} \frac{-2}{(x+h-2)(x-2)} = \frac{-2}{(x-2)(x-2)} = \frac{-2}{(x-2)^2}$$

$$\boxed{g'(x) = \frac{-2}{(x-2)^2}}$$

$(\sqrt{x+h+1})(\sqrt{x+h+1}) = (\sqrt{x+h+1})^2$

$= x+h+1$

(c) $k(x) = \sqrt{(x)+1}$

$$k'(x) = \lim_{h\to 0} \frac{k(x+h) - k(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)+1} - \sqrt{x+1}}{h} \cdot \frac{(\sqrt{x+h+1} + \sqrt{x+1})}{(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \lim_{h\to 0} \frac{(\cancel{x}+h+\cancel{1}) - (\cancel{x}+\cancel{1})}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \lim_{h\to 0} \frac{\cancel{h}\ 1}{\cancel{h}(\sqrt{x+h+1} + \sqrt{x+1})}$$

$$= \frac{1}{\sqrt{x+1} + \sqrt{x+1}}$$

$$= \frac{1}{2\sqrt{x+1}}$$

$$\boxed{k'(x) = \frac{1}{2\sqrt{x+1}}}$$

22. Given $f(x) = 2x^2 + 4$, find

(a) the average rate of change of $f(x)$ when x changes from 0 to 2.

x_1 x_2

$f(2) = 2(2)^2 + 4$
$= 12$
$f(0) = 0 + 4 = 4$

$$\text{AROC} = \frac{\Delta y}{\Delta x} = \frac{f(2) - f(0)}{2 - 0} = \frac{12 - 4}{2 - 0} = \frac{8}{2} = \boxed{4}$$

Pts:
$(0, f(0))$
$(2, f(2))$

(b) the instantaneous rate of change of $f(x)$ at $x = 0$.

$f'(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{[2(x+h)^2 + 4] - [2x^2 + 4]}{h}$$
$$= \lim_{h\to 0} \frac{[2(x^2 + 2xh + h^2) + 4] - 2x^2 - 4}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{+4} \cancel{-2x^2} \cancel{-4}}{h}$$
$$= \lim_{h\to 0} \frac{4xh + 2h^2}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(4x + 2h)}{\cancel{h}} = 4x$$

$f'(x) = 4x$

$f'(0) = 4(0) = \boxed{0}$

(c) The equation of the tangent line to $f(x)$ at $x = 0$.

$m_{tan} = ?$

$pt = ?$

$m_{tan} = f'(0) = 0$

$pt: (0, f(0)) = (0, 4)$

$y - y_1 = m(x - x_1)$
$y - 4 = 0(x - 0)$
$y - 4 = 0$
$\boxed{y = 4}$

23.

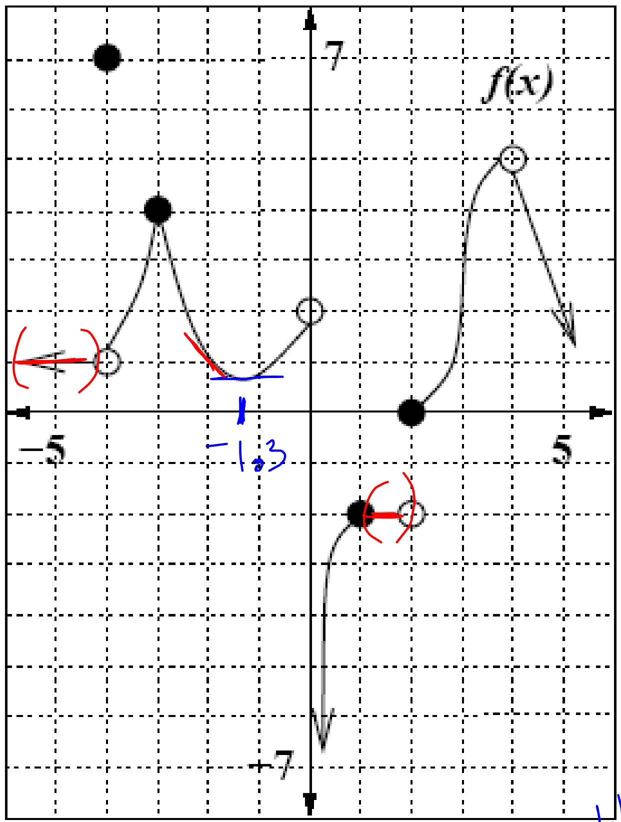
(a) Where is $f(x)$ discontinuous? Justify your answer using the definition of continuity.

$x = -4$ (limit value $\neq$ func. value)

$x = 0$ (no func. value) or no lim value

$x = 2$ (no limit value

$x = 4$ (no func. value)



(b) Where is $f(x)$ not differentiable? Justify your answer.

no deriv. exists

$x = -4, 0, 2, 4$ (not continuous)

$x = -3, 1$ (sharp turn, corner)

$x = 3$ (vert. tangent)

(c) Where is the instantaneous rate of change of $f(x)$ zero?

$f'(x) = m_{tan}$

$m_{tan} = 0 \longrightarrow$ horiz. tan. line

$\approx x = -1.3$

every x in the interval $(1, 2)$

every x in the interval $(-\infty, -4)$

not the endpts since not differentiable there