Week-In-Review #6

1. Find derivatives of the following:

(a) \( f(x) = x^2 + 7x^3 - \pi x' + 6 \)

\[
\begin{align*}
\frac{df}{dx} &= 2x + 21x^2 - \pi \\
\frac{d^2f}{dx^2} &= 2x + 21 \times 2 - \pi 
\end{align*}
\]

(b) \( g(x) = \sqrt[3]{x^3} - \frac{1}{x^3} + ex = x^{3/4} - x^{-3} + ex \)

\[
\begin{align*}
\frac{dg}{dx} &= \frac{3}{4}x^{-1/4} + 3x^{-4} + e
\end{align*}
\]

(c) \( h(x) = \frac{4x^3 - 7x + 2}{x^4} = \frac{4x^3}{x^4} - \frac{7x}{x^4} + \frac{2}{x^4} = 4x^{-1} - 7x^{-3} + 2x^{-4} \)

\[
\begin{align*}
\frac{dh}{dx} &= -4x^{-2} + 21x^{-4} - 8x^{-5}
\end{align*}
\]

(d) \( r(x) = 5x^2 \left( 2\sqrt{x} + \frac{3}{x^5} - 7 \right) = (5x^2)(2x^{1/2} + 3x^{-5} - 7) = 10x^{5/2} + 15x^{-3} - 35x^2 \)

\[
\begin{align*}
\frac{dr}{dx} &= 25x^{3/2} - 45x^{-4} - 70x
\end{align*}
\]
(e) \[ k(x) = (7x^4 - 10x + \sqrt{x}) \left( \frac{1}{x} - \frac{2}{3\sqrt{x}} + 8 \right) \]

\[ = \left(7x^4 - 10x + x^{1/2}\right) \left(\frac{1}{x} - \frac{2}{3\sqrt{x}} + 8\right) \]

**PRODUCT RULE**

\[ k'(x) = \left(28x^3 - 10 + \frac{1}{2}x^{-1/2}\right) \left(\frac{1}{x} - \frac{2}{3\sqrt{x}} + 8\right) + \left(7x^4 - 10x + x^{1/2}\right) \left(-1x^{-2} + \frac{2}{3} x^{-4/3}\right) \]

(f) \[ y = \frac{5x^4 + 5\sqrt{x^8} - 2x}{3x^6 - 9} = \frac{(5x^4 + x^{8/5} - 2x)}{(3x^6 - 9)} \]

**QUOTIENT RULE**

\[ y' = \frac{(3x^6 - 9)(20x^3 + \frac{8}{5} x^{3/5} - 2) - (5x^4 + x^{8/5} - 2x)(18x^5)}{(3x^6 - 9)^2} \]
(g) \( m(x) = 3(8 - 4x^2)^3 \)

\[
m'(x) = 9(8 - 4x^2)^2(-8x) = -72x(8 - 4x^2)^2
\]

(h) \( g(x) = -2\sqrt[5]{(x^2 + 4)^2} = -2(x^2 + 4)^{\frac{2}{5}} \)

\[
g'(x) = -\frac{4}{5}(x^2 + 4)^{-\frac{3}{5}}(2x) = \frac{-8x}{5(x^2 + 4)^{\frac{3}{5}}}
\]

(i) \( n(x) = \frac{10}{\sqrt{7 + x}} = \frac{10}{(7 + x)^{\frac{1}{2}}} \)

QUO. RULE:

\[
n'(x) = \left(\frac{7 + x}{(7 + x)^{\frac{1}{2}}}\right)' = \frac{1}{2}(7 + x)^{-\frac{1}{2}}(1)
\]

\[
n'(x) = \frac{-5(7 + x)\sqrt{7 + x}}{(7 + x)^{\frac{3}{2}}} = -5\left(\frac{1}{(7 + x)\sqrt{7 + x}}\right)
\]
(i) \[ y = (3x - 4x^7)^2 (2\sqrt{x} - 6)^3 \]

**PRODUCT RULE:**

\[ y' = \left[ 2(3x - 4x^7)(3 - 28x^6) \right] \left( 2x^{\frac{1}{2}} - 6 \right)^3 + \\
+ \left[ 2(3x - 4x^7)^2 \left( 3 \left( 2x^{\frac{1}{2}} - 6 \right)^2 \left( x^{-\frac{1}{2}} \right) \right) \right] \]
(k) \( f(x) = \frac{(2x^8 - 6x^5 + 1)^4}{(x - 7)(x + 3)^2} \)

QUOTIENT RULE:

\[
f'(x) = \left[ (x-7)(x+3)^2 \right] \left[ a \cdot 2 \cdot (x+3)^3 \right] - \left( 2x^8 - 6x^5 + 1 \right)^4 \left\{ (1)(x+3)^2 + (x-7) \cdot 2 \cdot (x+3)'(1) \right\} \]

\[
\left[ (x-7)(x+3)^2 \right]^2
\]
(1) \[ y = \frac{3}{4} (2x + 5)^4 (3x^2 - 7)^6 \]

\[ y' = \frac{1}{3} \left[ (2x+5)^4 (3x^2-7)^6 \right]^{-2/3} \left\{ 4(2x+5)^3 (3x^2-7)^6 + (2x+5)^4 \left[ 6 \left( 3x^2-7 \right) (6x) \right] \right\} \]

\[ \text{Chain Rule:} \]

\[ y' = [\frac{4}{3} (2x+5)^{4/3} (3x^2-7)^{6/3}] + (2x+5)^{4/3} \left[ \frac{6}{3} \left( 3x^2-7 \right)^{3/3} (6x) \right] \]

\[ \text{Rewrite } y \text{ as } y = (2x+5)^{4/3} (3x^2-7)^{6/3} \text{ and use prod. rule} \]
2. Find the value(s) of $x$ where $f(x) = \frac{x}{x^2 + 4}$ has a horizontal tangent line. 

\[ m_{\text{tan}} = 0 \implies f'(x) = 0 \]

**Quotient Rule**

\[
\frac{d}{dx} \left( \frac{x}{x^2 + 4} \right) = \frac{(x^2 + 4)(1) - x(2x)}{(x^2 + 4)^2}
\]

\[
= \frac{x^2 + 4 - 2x^2}{(x^2 + 4)^2}
\]

\[
f'(x) = \frac{-x^2 + 4}{(x^2 + 4)^2}
\]

\[
1(-x^2 + 4) = 0 (x^2 + 4)^2
\]

\[
-x^2 + 4 = 0
\]

\[
x^2 = 4 \implies x = \pm 2
\]
3. Find the equation of the tangent line to the following functions at the indicated point.

(a) \( f(x) = 5\sqrt{6x^2 + 4} \) at \( x = 0 \)

\[ f(0) = 5\sqrt{0+4} = 5(2) = 10 \]

\( f(x) = 5\sqrt{6x^2 + 4} \)

Point: \( (0, f(0)) = (0, 10) \)

Slope: \( f'(x) = \frac{5}{2} (6x^2 + 4)^{-\frac{1}{2}} \cdot (12x) \)

\[ f'(0) = \frac{5}{2} (4)^{-\frac{1}{2}} (0) = 0 = m_{\text{tan}} \]

Eqn of Line: \( y - y_1 = m(x - x_1) \)

\[ y - 10 = 0 (x - 0) \rightarrow y - 10 = 0 \rightarrow y = 10 \]

(b) \( g(x) = (x^2)(x - 1)^3 \) at \( (2, 4) = P_1 \)

Slope: \( g'(x) = (2x)(x - 1)^3 + (x^2)[3(x-1)^2] \)

\[ g'(2) = (4)(1)^3 + 4[3(1)^2(1)] \]

\[ = 4 + 12 = 16 = m \]

Eqn: \( y - y_1 = m(x - x_1) \)

\[ y - 4 = 16 (x - 2) \rightarrow y - 4 = 16x - 32 \rightarrow y = 16x - 28 \]
4. A ball is thrown vertically upward from the ground at a velocity of 64 ft/sec. Its distance from the ground at $t$ seconds is given by $s(t) = -16t^2 + 64t$.

(a) Find the velocity function, $v(t)$.

$$v(t) = s'(t) = -32t + 64$$

(b) How fast is the ball moving 2 seconds after being thrown? 3 seconds after being thrown?

$$v(2) = -32(2) + 64 = 0 \text{ ft/sec}$$

$$v(3) = -32(3) + 64 = -32 \text{ ft/sec} \rightarrow \text{falling 32 ft/sec}$$

(c) How long after the ball is thrown does it reach its maximum height?

$$-32t + 64 = 0$$

$$32t = 64$$

$$t = 2 \text{ sec}$$

$$v(t) = 0$$

(d) How high will the ball go?

$$s(2) = -16(2)^2 + 64(2) = 64 \text{ ft}$$

(e) When does the ball hit the ground?

$$s(t) = 0$$

$$-16t^2 + 64t = 0$$

$$(-16t)(t - 4) = 0$$

$$-16t = 0 \quad t = 0 \quad t = 4$$

$$4 \text{ sec after it's thrown}$$
5. A particular book is sold at a price according to the price-demand function, \( p(x) = -0.5x + 50 \). It costs a total of $650 to produce 75 of these books and there are $275 of fixed costs associated with producing the books. If \( x \) represents the number of these books that are made and sold, find the following.

(a) \( C(x) \), \( R(x) \), and \( P(x) \)

\[
C(x) = mx + b = mx + 275
\]

\[
650 = m(75) + 275
\]

\[
375 = 75m
\]

\[
S = m
\]

\[
C(x) = 5x + 275
\]

\[
R(x) = (\text{price} \times \text{quantity})
\]

\[
= (-0.5x + 50)(x)
\]

\[
R(x) = -0.5x^2 + 50x
\]

\[
P(x) = R(x) - C(x)
\]

\[
= (-0.5x^2 + 50x) - (5x + 275)
\]

\[
P(x) = -0.5x^2 + 45x - 275
\]

(b) Marginal cost, marginal revenue, and marginal profit functions

\[
M(x) = C'(x) = 5
\]

\[
MR(x) = R'(x) = -x + 50
\]

\[
MP(x) = P'(x) = -x + 45
\]

(c) \( R'(10) \) and interpret.

\[
R'(10) = -10 + 50 = 40
\]

When 10 books are sold, rev. is inc. at a rate of $40/book.

\[
\Rightarrow \text{The approx. revenue from the sale of the 1st book is $40}
\]
6. The total cost (in dollars) of producing \( x \) items is given by \( C(x) = 500 + 40x + 0.05x^2 \).

(a) Find the average cost function.
\[
AC(x) = \frac{C(x)}{x} = \frac{500 + 40x + 0.05x^2}{x} = \frac{500x^{-1}}{x} + 40 + 0.05x
\]

(b) Find the marginal cost function.
\[
MC(x) = C'(x) = 40 + 0.1x
\]

(c) Find the marginal average cost function.
\[
AC'(x) = -500x^{-2} + 0.05
\]

(d) Find the average cost per unit if 20 items are produced.
\[
AC(20) = \$66
\]

(e) Find the marginal average cost at a production level of 20 items and interpret.
\[
AC'(20) = -1.2 \rightarrow \text{At a prod. level of 20 items, the avg cost per item is dec. at a rate of } \$1.20/\text{item}
\]

(f) Estimate the average cost per item (using the information found above) if 21 items are produced.
\[
66 - 1.20 = \$64.80/\text{item}
\]