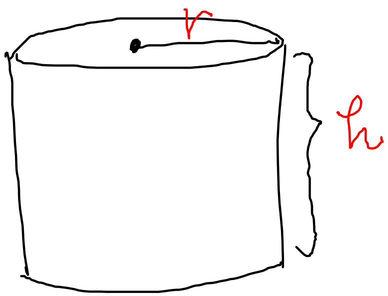
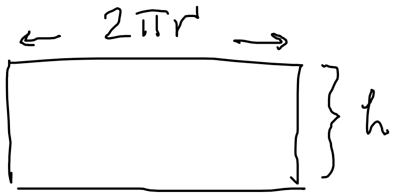
Week-In-Review #15

1. Find the equations of the volume and surface area of a cylinder as functions of its radius and its height.



$$V = (\pi r^2) h$$

$$SA = \underbrace{2\pi r^2}_{\text{top + bottom}} + 2\pi r h$$



2. Determine the domain of the following functions:

(a) $f(x, y) = e^{3x} - y \ln \boxed{(x + 5)}$

$x + 5 > 0$

$x > -5$

$\boxed{\text{All } (x, y) \text{ such that } x > -5}$

(b) $g(x, y) = \dfrac{6}{\boxed{2x + 3y}}$

$2x + 3y \neq 0$

$\boxed{\text{All } (x, y) \text{ such that } 2x + 3y \neq 0}$

3. A company sells cameras and film to be used in the camera. Let x be the number of cameras sold per day and y be the number of rolls of film sold per day. Suppose the price of each roll of film is given by $q = 802 - 3x - 0.5y$ and the price of each camera is given by $p = 1400 - 12x - y$. Also, suppose the daily cost function for the company is given to be $C(x, y) = 15000 + 50x + 0.5y$.

(a) Find the revenue function for the company.

(b) Find the profit function for the company.

(c) What is the profit/loss for the company when 5 cameras (x) and 50 rolls (y) of film are sold in one day?

(a) Rev = (price)(quantity)

$= p \cdot x + q \cdot y$ (rev from cameras + rev from film)

$= (1400 - 12x - y)x + (802 - 3x - 0.5y)y$

$R(x,y) = 1400x - 12x^2 - xy + 802y - 3xy - 0.5y^2$

$$\boxed{R(x,y) = -12x^2 - 0.5y^2 + 1400x + 802y - 4xy}$$

(b) $P = R - C = (-12x^2 - 0.5y^2 + 1400x + 802y - 4xy) - (15000 + 50x + 0.5y)$

$$\boxed{P(x,y) = -12x^2 - 0.5y^2 + 1350x + 801.5y - 4xy - 15000}$$

(c) $$\boxed{P(5,50) = \$29,275}$$

4. Find $f_x(x, y)$ and $f_y(x, y)$ for the following.

$f_x = \frac{\partial f}{\partial x}$ $f_y = \frac{\partial f}{\partial y}$

(a) $f(x, y) = 2x^2 - 3y^2 + (5x^2)(y^2) - e$

$(5x^2)' = 5(2x) = 10x$

$f_x = 4x - 0 + 10xy^2 - 0$

$f_y = 0 - 6y + 5x^2(2y) - 0$

$f_y = -6y + 10x^2y$

(b) $f(x, y) = \boxed{(7y - 3x^2)}^4$

$f_x = 4(7y - 3x^2)^3(0 - 6x)$

$f_y = 4(7y - 3x^2)^3(7 - 0)$

(c) $f(x,y) = \dfrac{x^2}{y^4+5}$

$$f_x = \frac{(y^4+5)(2x) - \cancel{x^2(0)}}{(y^4+5)^2}$$

$$f_x = \frac{2x}{y^4+5}$$

$$f_y = \frac{\cancel{(y^4+5)(0)} - x^2(4y^3)}{(y^4+5)^2}$$

$$f_y = \frac{-4x^2y^3}{(y^4+5)^2}$$

(d) $f(x,y) = 2e^{xy} + e^{x-y} = 2e^{\boxed{xy}} + e^{\boxed{x-y}}$

$$f_x = 2e^{xy}(y) + e^{x-y}(1)$$

$$f_y = 2e^{xy}(x) + e^{x-y}(-1)$$

(e) $f(x,y) = 4\ln\boxed{(x^2+3x+y^7)} - \pi$

$$f_x = 4\left(\frac{1}{x^2+3x+y^7}\right)(2x+3)$$

$$f_x = \frac{4(2x+3)}{x^2+3x+y^7}$$

$$f_y = 4\left(\frac{1}{x^2+3x+y^7}\right)(7y^6)$$

$$f_y = \frac{28y^6}{x^2+3x+y^7}$$

5. A company has a monthly production function given by

$$f(x, y) = 100x^{0.6}y^{0.4}$$

where x is the amount of labor used and y is the amount of capital used.

(a) If the company is now using 15 units of labor and 23 units of capital, find the marginal productivity of labor and the marginal productivity of capital.

Marg. prod. of labor : $f_x = 100y^{0.4}(0.6x^{-0.4}) = 60x^{-0.4}y^{0.4}$

$f_x(15,23) = 60(15)^{-0.4}(23)^{0.4} \approx 71.19$

Marg. prod. of capital : $f_y = 100x^{0.6}(0.4y^{-0.6}) = 40x^{0.6}y^{-0.6}$

$f_y(15,23) = 40(15)^{0.6}(23)^{-0.6} \approx 30.95$

(b) For the greatest increase in the company's productivity, should labor or capital be increased from its current state?

labor

$(71 > 31)$

6. Find all second-order partial derivatives: f_{xx}, f_{yy}, f_{xy}, and f_{yx}.

$(f_x)_x \quad (f_y)_y \quad (f_x)_y \quad (f_y)_x$

(a) $f(x, y) = (5x)y^3 - (7x^3)y$

$f_x = 5y^3 - 21x^2y$

$f_{xx} = 0 - 42xy$

$f_{xy} = 15y^2 - 21x^2(1)$

$f_y = 5x(3y^2) - 7x^3(1)$

$f_y = 15xy^2 - 7x^3$

$f_{yy} = 15x(2y) - 0 = 30xy$

$f_{yx} = 15y^2(1) - 21x^2$

$\overset{?}{=}$ ✓

(b) $f(x, y) = \dfrac{x^2}{y} = x^2y^{-1}$

$f_x = 2xy^{-1}$

$f_{xx} = 2y^{-1}$

$f_{xy} = 2x(-1y^{-2})$

$= -2xy^{-2}$

$f_y = x^2(-1y^{-2}) = -x^2y^{-2}$

$f_{yy} = -x^2(-2y^{-3}) = 2x^2y^{-3}$

$f_{yx} = -y^{-2}(2x) = -2xy^{-2}$

$=$ ✓

(c) $f(x,y) = xe^{xy} = (x)(e^{\boxed{xy}})$

prod. rule

$f_x = 1e^{xy} + x[e^{xy}(y)]$

$f_x = e^{\boxed{xy}} + (xy)e^{xy}$

$f_{xx} = e^{xy}(y) + \{ye^{xy} + xy[e^{xy}(y)]\}$

$f_{xy} = e^{xy}(x) + \{xe^{xy} + xy[e^{xy}(x)]\}$

$2xe^{xy} + x^2ye^{xy}$

$\Leftrightarrow = ✓$ $f_{yx} = 2xe^{xy} + x^2[e^{xy}(y)]$

$f_y = x[e^{xy}(x)]$

$f_y = (x^2)e^{\boxed{xy}}$

$f_{yy} = x^2[e^{xy}(x)]$

7. Find all of the critical points for $f(x, y) = 3x^2 + 5y^2 - 8xy + 2x + 6y + 3$

* $f_x = 0$ AND $f_y = 0$ simultaneously *

$f_x = 6x - 8y + 2 = 0$ $\qquad$ $f_y = 10y - 8x + 6 = 0$

$4(6x - 8y + 2 = 0)$

$3(-8x + 10y + 6 = 0)$

$\rightarrow$

$\cancel{24x} - 32y + 8 = 0$

$\oplus\ \cancel{-24x} + 30y + 18 = 0$

$-2y + 26 = 0$

$2y = 26$

$y = 13$

$\rightarrow 6(x) - 8(13) + 2 = 0$

$6x = 102$

$x = 17$

$\boxed{CP: (17, 13)}$

8. Locate any critical points of the following and, if possible, identify each as local extrema or a saddle point.

(a) $f(x, y) = 2x^2 + 5y^2 + 6x - 2y + 12$

$\longrightarrow f_x = 4x + 6$ $\qquad \longrightarrow f_y = 10y - 2$

$f_{xx} = 4$ $\qquad f_{yy} = 10$

$f_{xy} = 0$ $\qquad f_{yx} = 0$

CP: $f_x = 0$; $f_y = 0$

$4x + 6 = 0$ $\qquad 10y - 2 = 0$

$4x = -6$ $\qquad 10y = 2$

$x = -3/2$ $\qquad y = 1/5$

$\boxed{\text{CP } \left(-3/2, 1/5\right)}$

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$ evaluated at CP

$= (4)(10) - (0)^2$

$= 40 > 0 \longrightarrow$ Either a local max or min $\Longrightarrow$ need to look @ f_{xx} to figure out which you have

$f_{xx} = 4 > 0 \Longrightarrow$ $\boxed{\text{CP is a local min}}$

(b) $f(x, y) = -3x^2 + xy - y^2 - 4x - 3y$

$f_x = -6x + y - 4$ $\quad$ $f_y = x - 2y - 3$

$f_{xx} = -6$ $\quad$ $f_{yy} = -2$

$f_{xy} = 1$ $\quad$ $f_{yx} = 1$

CP: $2(-6x + y - 4 = 0)$

$x - 2y - 3 = 0$

$$\begin{aligned} -12x + \cancel{2y} - 8 &= 0 \\ + \quad x - \cancel{2y} - 3 &= 0 \\ \hline -11x - 11 &= 0 \\ -11x &= 11 \\ x &= -1 \end{aligned}$$

$-6(-1) + y - 4 = 0$

$6 + y - 4 = 0$

$y = -2$

$\boxed{\text{CP: } (-1, -2)}$

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$= (-6)(-2) - (1)^2$

$= 12 - 1$

$= 11 > 0$

$f_{xx} = -6 < 0$

$\Downarrow$

$\boxed{\text{CP is a local max}}$

(c) $f(x,y) = xy - x^3 - y^2$

$f_x = y - 3x^2$

$f_{xx} = -6x$

$f_{xy} = 1$

$f_y = x - 2y$

$f_{yy} = -2$

$f_{yx} = 1$

CP:

$y - 3x^2 = 0$

$y = 3x^2$

$x - 2y = 0$

$x - 2(3x^2) = 0$

$x - 6x^2 = 0$

$x(1-6x) = 0$

$x = 0$ or $1 - 6x = 0$

$1 = 6x$

$1/6 = x$

$x = 0 \rightarrow y = 3(0)^2 = 0$

$\boxed{CP: (0,0)}$

$x = 1/6 \rightarrow y = 3\left(\frac{1}{6}\right)^2 = 3/36 = 1/12$

$\boxed{CP: (1/6, 1/12)}$

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (-6x)(-2) - (1)^2$

$D = 12x - 1$

$D(0,0) = 12(0) - 1 = -1 < 0$

$\boxed{(0,0) \text{ is a saddle pt}}$

$D\left(\frac{1}{6}, \frac{1}{12}\right) = 12\left(\frac{1}{6}\right) - 1 = 2 - 1 = 1 > 0$

$f_{xx}\left(\frac{1}{6}, \frac{1}{12}\right) = -6\left(\frac{1}{6}\right) = -1 < 0$

$\boxed{\left(\frac{1}{6}, \frac{1}{12}\right) \text{ is a local max}}$

9. A firm manufactures and sells two products, X and Y, that sell for \$15 and \$10 each, respectively. The cost of producing x units of X and y units of Y is

$$C(x, y) = 400 + 7x + 4y + 0.01(3x^2 + xy + 3y^2)$$

Find the values of x and y that maximize the firm's $\boxed{\text{profit.}}$ $\Rightarrow$ $\boxed{x = 120, y = 80}$

$$\text{Profit} = R - C = (15x + 10y) - [400 + 7x + 4y + 0.03x^2 + 0.01xy + 0.03y^2]$$

$$P(x,y) = -0.03x^2 - 0.03y^2 + 8x + 6y - 0.01xy - 400$$

$$P_x = -0.06x + 8 - 0.01y \qquad P_y = -0.06y + 6 - 0.01x$$

$$P_{xx} = -0.06 \qquad P_{yy} = -0.06$$

$$P_{xy} = -0.01 \qquad P_{yx} = -0.01$$

CP: $-0.06x - 0.01y + 8 = 0$

$-6(-0.01x - 0.06y + 6 = 0)$

$$-0.06x - 0.01y + 8 = 0$$
$$+\ 0.06x + 0.36y - 36 = 0$$
$$0.35y - 28 = 0$$
$$0.35y = 28$$
$$y = 80$$

$$\to -0.06x - 0.01(80) + 8 = 0$$
$$x = 120$$

$\boxed{\text{CP: } (120, 80)}$

$$D = P_{xx}P_{yy} - (P_{xy})^2 = (-0.06)(-0.06) - (-0.01)^2$$
$$= +$$

$\to P_{xx} = - \to$ local max @ CP

10. Continental Airlines has the restriction that a carry-on bag have a maximum combined linear measurement (L+W+H) of 51 inches. What are the dimensions of the largest bag, in volume, that will meet this requirement?

max $V = LWH$

know $L+W+H=51 \rightarrow H = 51-L-W$

$V = LW(51-L-W)$

$V = 51LW - L^2W - LW^2$

$V_L = 51W - 2LW - W^2$

$V_{LL} = -2W$

$V_{LW} = 51 - 2L - 2W$

$V_W = 51L - L^2 - 2LW$

$V_{WW} = -2L$

$V_{WL} = 51 - 2L - 2W$

CP: $51W - 2LW - W^2 = 0$ AND $51L - L^2 - 2LW = 0$

$W(51-2L-W) = 0$

$\rightarrow \cancel{W=0}$ or $51-2L-W=0$

no volume

$L(51-L-2W) = 0$

$\rightarrow \cancel{L=0}$ or $51-L-2W=0$

no volume

$$51-2L-W=0$$
$$-2(51-L-2W=0)$$

$$51-\cancel{2L}-W=0$$
$$\oplus\ -2(51)+\cancel{2L}+4W=0$$
$$-51+3W=0$$
$$3W=51$$
$$W=17$$

$$51-2L-17=0$$
$$-2L+34=0$$
$$2L=34$$
$$L=17$$

$$\boxed{CP: (17,17)}$$

$$D=V_{LL}V_{WW}-(V_{LW})^2$$
$$=(-2W)(-2L)-(51-2L-2W)^2$$

$$D(17,17)=867>0 \rightarrow V_{LL}=-2W$$
$$V_{LL}(17,17)=-2(17)=-$$
$\Rightarrow$ CP is a max

Dimensions: $L=17''$
$W=17''$

$H=51-L-W$
$=51-17-17$
$H=17''$