Exam III: Night Before Drill

1. How long will it take for an investment earning 5.4% interest, compounded continuously, to triple?

$A = Pe^{rt}$

$3\cancel{P_0} = \cancel{P_0} e^{0.054t}$

$\ln 3 = \ln e^{\boxed{0.054t}}$

$\ln 3 = 0.054t\,(\cancel{\ln e}^{1})$

$\frac{\ln 3}{0.054} = t$

$\boxed{t \approx 20.345 \text{ yrs}}$

2. Find $f[g(x)]$ and $g[f(x)]$ given

(a) $f(x) = e^{x+2}$; $\qquad g(x) = \sqrt{x+7}$

$f[g(x)] = f(\sqrt{x+7}) = e^{\sqrt{x+7}+2}$

$g[f(x)] = g(e^{x+2}) = \sqrt{e^{x+2}+7}$

(b) $f(x) = x^2 + 8$; $\qquad g(x) = \ln x$

$f[g(x)] = f(\ln x) = (\ln x)^2 + 8$

$g[f(x)] = g(x^2+8) = \ln(x^2+8)$

3. Find the interval(s) where $f(x) = e^{\boxed{x^3+2x^2-4x}}$ is $\underbrace{\text{increasing/decreasing}}_{f'(x)}$.

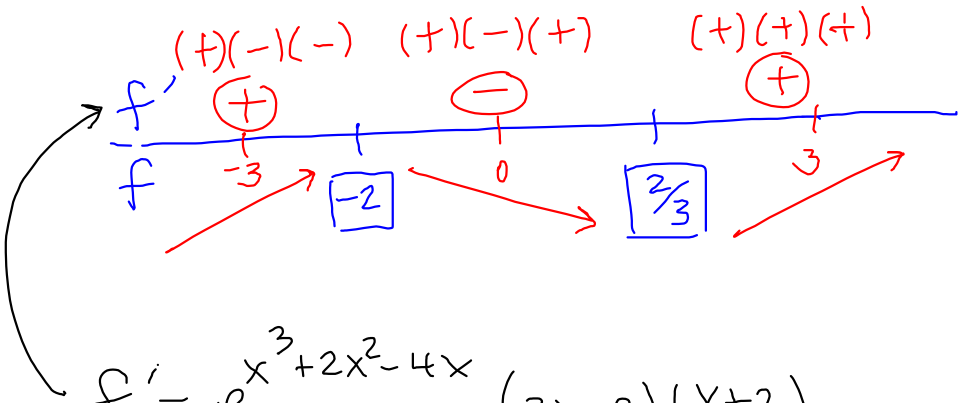
Domain of f : $\mathbb{R}$

$$f'(x) = e^{x^3+2x^2-4x}\left(3x^2+4x-4\right) = 0$$

$$e^{x^3+2x^2-4x} \neq 0 \qquad 3x^2+4x-4 = 0$$

$$(3x-2)(x+2) = 0$$

$$x = 2/3 \qquad x = -2 \quad \leftarrow \text{CV's}$$



$$f' = e^{x^3+2x^2-4x}(3x-2)(X+2)$$

$$\boxed{\begin{array}{l} \text{INC: } (-\infty,-2) \quad (2/3,\infty) \\ \text{DEC: } (-2,\ 2/3) \end{array}}$$

4. Find intervals, if any exist, where the following functions are $\underbrace{\text{concave up/concave down}}_{f''}$ and determine any points of inflection.

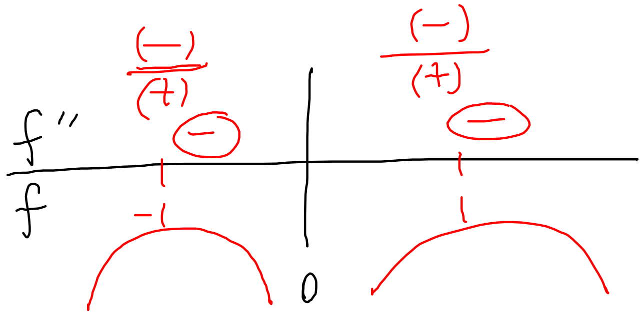
(a) $f(x) = \ln \boxed{x^2}$ D: $x^2 > 0 \Rightarrow x \neq 0$

$f' = \left(\frac{1}{x^2}\right)(2x) = \frac{2}{x} = 2x^{-1}$

$f'' = -2x^{-2} = \boxed{\frac{-2}{x^2}} = 0$

$\rightarrow -2 \neq 0$

f'' undef. @ $x=0$ (not in Domain)



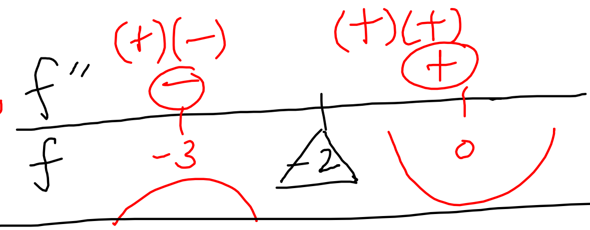
C↓: $(-\infty, 0)$ $(0, \infty)$
C↑: never
NO Inf. Pts.

(b) $f(x) = (5x)(e^x)$ D: $\mathbb{R}$

$f' = 5e^x + 5xe^x$
$= (5e^x)([1+x])$

$f'' = 5e^x(1+x) + 5e^x(1)$
$= 5e^x[(1+x)+1]$
$= \boxed{(5e^x)(x+2)}$

$f'' = 0 \rightarrow 5e^x \neq 0$ $x+2=0$
$x=-2$ possible I.P.



C↓: $(-\infty, -2)$ IP @ $x=-2$
C↑: $(-2, \infty)$ $(-2, f(-2))$

5. The cost of producing x units of a product is given by $C(x) = 800 + 200x - 100\ln x$ for $x \geq 1$. Find the minimum average cost. $\rightarrow \bar{C}(e^9) \approx \$1{,}620{,}516.79$

$$\text{Avg. cost} = \bar{C} = \frac{C}{x} = \frac{800 + 200x - 100\ln x}{x}$$

$$\bar{C}' = \frac{(x)\left[200 - 100\left(\frac{1}{x}\right)\right] - \left[800 + 200x - 100\ln x\right](1)}{x^2}$$

$$= \frac{\cancel{200x} - 100 - 800 - \cancel{200x} + 100\ln x}{x^2} = \frac{-900 + 100\ln x}{x^2}$$

$$\bar{C}' = 0 = \frac{-900 + 100\ln x}{x^2}$$

$$\Rightarrow -900 + 100\ln x = 0$$

$$100\ln x = 900$$

$$\ln x = 9$$

$$x = e^9$$

$\rightarrow$ only one CV in interval so will either be abs. max or abs. min.

$$\bar{C}'' = \frac{(x^2)\left(100\left(\frac{1}{x}\right)\right) - (-900 + 100\ln x)(2x)}{x^4}$$

$$= \frac{100x + 1800x - 200x\ln x}{x^4}$$

$$= \frac{1900\cancel{x} - 200\cancel{x}\ln x}{x^{\cancel{4}3}}$$

$$\bar{C}'' = \frac{1900 - 200\ln x}{x^3} \rightarrow \bar{C}''(e^9) = +$$

abs. min

6. Find $\frac{dy}{dx}$ given the following:

(a) $y = x^2 3^{\boxed{x^2+4x}} + 3^2$

$$\frac{dy}{dx} = (2x)\,3^{x^2+4x} + x^2\left[3^{x^2+4x}(2x+4)(\ln 3)\right] + 0$$

(b) $y = \log_{10}\boxed{(\ln(x^2+9))} + \ln 5$

$$\frac{dy}{dx} = \left(\frac{1}{\ln(x^2+9)}\right)\left[\left(\frac{1}{x^2+9}\right)(2x)\right]\left(\frac{1}{\ln 10}\right) + 0$$

(c) $y = e^u; u = -2x^2$

Method 1

$$\frac{dy}{du} = e^u$$

$$\frac{du}{dx} = -4x$$

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$$

$$= e^u(-4x) = \boxed{e^{-2x^2}(-4x)}$$

Method 2 $y = e^{\boxed{-2x^2}}$

$$\boxed{\frac{dy}{dx} = e^{-2x^2}(-4x)}$$

7. Given the price-demand equation $\overset{f(p)}{x} = 10(\boxed{p-9})^2$,

(a) Find the elasticity of demand, $E(p)$.

$$E(p) = \frac{-p f'(p)}{f(p)} = \frac{-p\{\cancel{20}^{2}(\cancel{p-9})(1)\}}{\cancel{10}(p-9)^{\cancel{2}\,1}} = \frac{-2p}{p-9}$$

(b) If $p = \$5$, is demand elastic, inelastic, or is there unit elasticity?

$$E(5) = \frac{-2(5)}{5-9} = \frac{-10}{-4} = \frac{10}{4} > 1 \Rightarrow \boxed{\text{elastic}}$$

(c) Should the price be raised, lowered, or kept at \$5 in order to increase revenue?

elastic $\longrightarrow$ lower the price

(d) At what price will revenue be $\underbrace{\text{maximized}}_{E(p)=1}$?

$$\frac{-2p}{p-9} = 1 \longrightarrow -2p = p-9 \quad -3p = -9 \longrightarrow \boxed{p = \$3}$$

8. Evaluate the following:

(a) $\int (2x + \frac{5}{x^3} - x\sqrt[4]{x} - e)\,dx$

$x^1 \cdot x^{1/4}$

$$\int (2x^1 + 5x^{-3} - x^{5/4} - e)\,dx$$

$$= \cancel{2}\left(\frac{x^2}{\cancel{2}}\right) + 5\left(\frac{x^{-2}}{-2}\right) - \frac{x^{9/4}}{(9/4)} - ex + C$$

$$= \boxed{x^2 - \frac{5}{2}x^{-2} - \frac{4}{9}x^{9/4} - ex + C}$$

$$\int e^x dx = e^x + C$$

$$\int \frac{1}{x} dx = \int x^{-1} dx = \ln|x| + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int k\,dx = kx + C$$

$$\frac{y^m}{y^n} = y^{m-n}$$

(b) $\int \frac{y^2 - \sqrt{y}}{y^3}\,dy = \int \left(\frac{y^2}{y^3} - \frac{y^{1/2}}{y^3}\right) dy = \int \left(\frac{1}{y} - y^{-5/2}\right) dy$

$$= \ln|y| - \frac{y^{-3/2}}{(-3/2)} + C$$

$$= \boxed{\ln|y| + \frac{2}{3}y^{-3/2} + C}$$

(c) $\int e^{\boxed{4t}}\, dt$

$u = 4t$

$du = 4\, dt$

$\frac{1}{4} du = dt$

$\int e^{u} \left(\frac{1}{4} du\right)$

$= \int \frac{1}{4} e^{u} du = \frac{1}{4} e^{u} + C \longrightarrow \boxed{\frac{1}{4} e^{4t} + C}$

(d) $\int \frac{3}{5x \cdot \boxed{\ln 4x}}\, dx$

$u = \ln \boxed{4x}$

$\frac{3}{5}(du) = \left(\frac{1}{\cancel{4}x}\right)(\cancel{4})\, dx = \left(\frac{1}{x} dx\right) \frac{3}{5}$

$\frac{3}{5} du = \frac{3}{5x} dx$

$\int \frac{\frac{3}{5} du}{u}$

$\int \frac{3}{5}\left(\frac{1}{u}\right) du = \frac{3}{5} \ln |u| + C \longrightarrow \boxed{\frac{3}{5} \ln |\ln 4x| + C}$

(e) $\int (10x-20)e^{\boxed{x^2-4x}}dx$

$u = x^2 - 4x$

$du = (2x-4)\,dx$

$5du = \boxed{(10x-20)\,dx}$

$\int 5\,e^{u}\,du$

$= 5e^{u} + C \rightarrow \boxed{5e^{x^2-4x} + C}$

(f) $\int t\sqrt{2t+5}\,dt = \int t\,\boxed{(2t+5)}^{1/2}\,dt$

$u = 2t+5$

$du = 2\,dt \rightarrow dt = \frac{1}{2}du$

$u-5 = 2t$

$\frac{u-5}{2} = t$

$\int \left(\frac{u-5}{2}\right) u^{1/2} \left(\frac{1}{2}du\right)$

$\int \frac{1}{4}(u-5)\,u^{1/2}\,du = \frac{1}{4}\int \left(u^{3/2} - 5u^{1/2}\right)du = \frac{1}{4}\left[\frac{u^{5/2}}{(5/2)} - \frac{5u^{3/2}}{(3/2)}\right] + C$

$\rightarrow \boxed{\frac{1}{4}\left[\frac{2}{5}(2t+5)^{5/2} - \frac{10}{3}(2t+5)^{3/2}\right] + C}$

(g) $\int \frac{y+2.5}{(y^2+5y+6)^3}\,dy$

$u = y^2+5y+6$

$du = (2y+5)\,dy$

$du = 2\boxed{(y+5/2)\,dy}$

$\frac{1}{2}du = (y+2.5)\,dy$

$\int \frac{\frac{1}{2}\,du}{u^3}$

$= \int \frac{1}{2} u^{-3}\,du$

$= \frac{1}{2}\,\frac{u^{-2}}{-2} + C \longrightarrow \boxed{-\frac{1}{4}(y^2+5y+6)^{-2} + C}$

9. Find the cost function for a tape manufacturer, if the marginal cost, in dollars/case, is given by $150 - 0.1e^x$, where x is the number of cases of tape produced and the manufacturer has \$100 worth of fixed costs.

$C(0) = \$100$

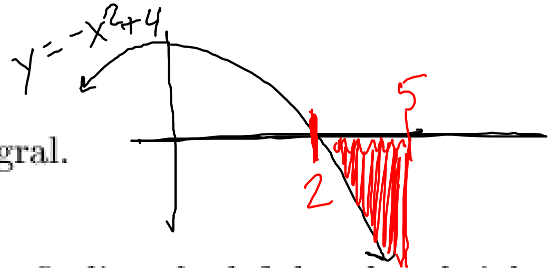
$C = \int C'\,dx = \int (150 - 0.1e^x)\,dx = 150x - 0.1e^x + K$

$C(0) = \cancel{150(0)}^{0} - 0.1\cancel{e^0}^{1} + K = 100$

$-0.1 + K = 100 \longrightarrow K = 100.1$

$\longrightarrow \boxed{C(x) = 150x - 0.1e^x + 100.1}$

10. Given $\int_2^5 (-x^2+4)\,dx$



(a) Sketch the region indicated by this integral.

$-x^2+4=0$

$x^2=4 \rightarrow x=\pm 2$

(b) Approximate the value of the integral by finding the left hand and right hand Riemann sums with 4 rectangles and then again with $\boxed{\text{100 rectangles}}$. Draw pictures of the areas being found with 4 rectangles.

$\rightarrow$ Riemann $y_1=-x^2+4$

$a=2,\ b=5$

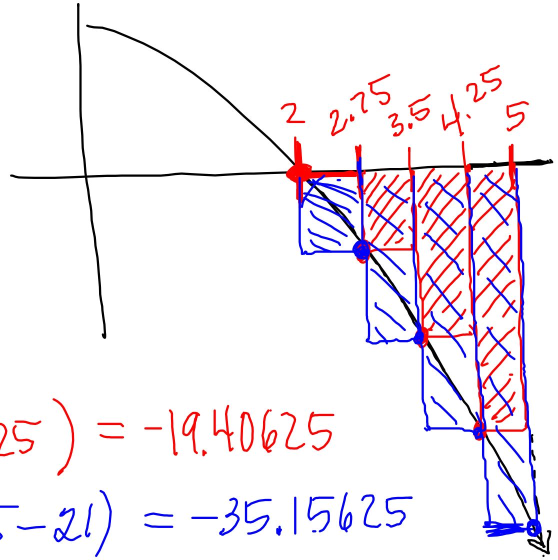
$n=100$

$L_{100}=-26.68545$

$R_{100}=-27.31545$

width of rect $=\Delta x=\frac{b-a}{n}=\frac{5-2}{4}=\frac{3}{4}$

x	$y=-x^2+4$
2	0
2.75	-3.5625
3.5	-8.25
4.25	-14.0625
5	-21



$L_4 = LHS = \frac{3}{4}(0-3.5625-8.25-14.0625) = -19.40625$

$R_4 = RHS = \frac{3}{4}(-3.5625-8.25-14.0625-21) = -35.15625$

(c) Find the exact value of the integral.

$\text{fnInt}(-x^2+4, x, 2, 5) = \boxed{-27}$

(d) How much area is ~~there between the curve~~ $y=-x^2+4$ and the x-axis from $x=2$ to $x=5$?

$\boxed{27 \text{ Sq. units}}$

11. Evaluate the following EXACTLY:

(a) $\int_a^b (x^3 + \frac{1}{x})\, dx$ $\quad (0 < a < b)$

$$= \frac{x^4}{4} + \ln|x| \Big|_a^b$$

$$= \left(\frac{b^4}{4} + \ln|b| \right) - \left(\frac{a^4}{4} + \ln|a| \right)$$

$$= \frac{b^4}{4} - \frac{a^4}{4} + \underbrace{\ln|b| - \ln|a|} = \frac{b^4}{4} - \frac{a^4}{4} + \ln\left|\frac{b}{a}\right|$$

(b) $\int_2^4 (e^t + \pi)\, dt$ $= e^t + \pi t \Big|_2^4$

$$= (e^4 + 4\pi) - (e^2 + 2\pi) = \boxed{e^4 - e^2 + 2\pi}$$

(c) $\displaystyle\int_{x=1}^{x=3} \frac{x^2+1}{x^3+3x}\,dx$

$u = x^3 + 3x \implies$

$du = (3x^2+3)\,dx$

$du = 3\boxed{(x^2+1)\,dx}$

$\frac{1}{3}du = (x^2+1)\,dx$

$x = 1 \longrightarrow u = 1^3 + 3(1) = 4$

$x = 3 \longrightarrow u = 3^3 + 3(3) = 36$

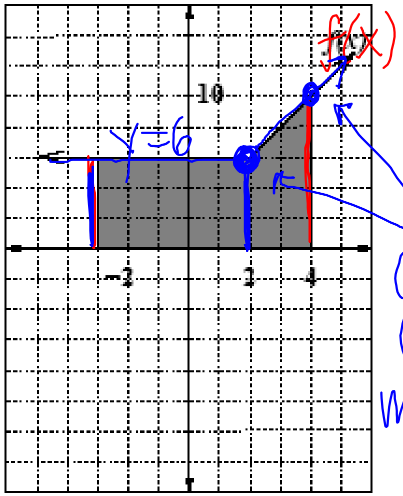
$\displaystyle\int_{u=4}^{u=36} \frac{\frac{1}{3}du}{u}$

$$\int_4^{36} \frac{1}{3}\left(\frac{1}{u}\right)du = \frac{1}{3}\ln|u| \Big|_4^{36} = \frac{1}{3}\ln|36| - \frac{1}{3}\ln|4|$$

$$= \frac{1}{3}\left[\ln 36 - \ln 4\right]$$

$$= \frac{1}{3}\ln\left(\frac{36}{4}\right) = \boxed{\frac{1}{3}\ln 9}$$

12. Write a definite integral to indicate the shaded area in the graph below.



$(2, 6)$

$(4, 10)$

$$\text{Area} = \int_{-3}^{4} f(x)\,dx$$

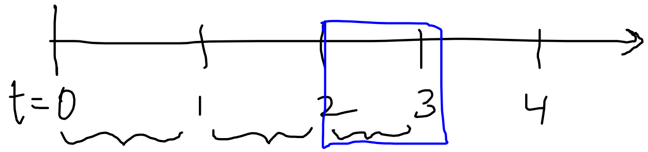
$$= \int_{-3}^{2} 6\,dx + \int_{2}^{4} (2x+2)\,dx$$

$m = \frac{10-6}{4-2} = \frac{4}{2} = 2$

$y - 6 = 2(x-2)$

$y - 6 = 2x - 4 \implies y = 2x + 2$

13. Suppose copper is being extracted from a mine at a rate given by $y = 100e^{-0.2t}$, where t is the number of years since mining began and y is measured in tons of copper/year. At this rate, how much copper will be extracted during the third year of mining?



$$\int_2^3 100e^{-0.2t}\,dt \to \text{fnInt}(100e^{-0.2t}, t, 2, 3)$$

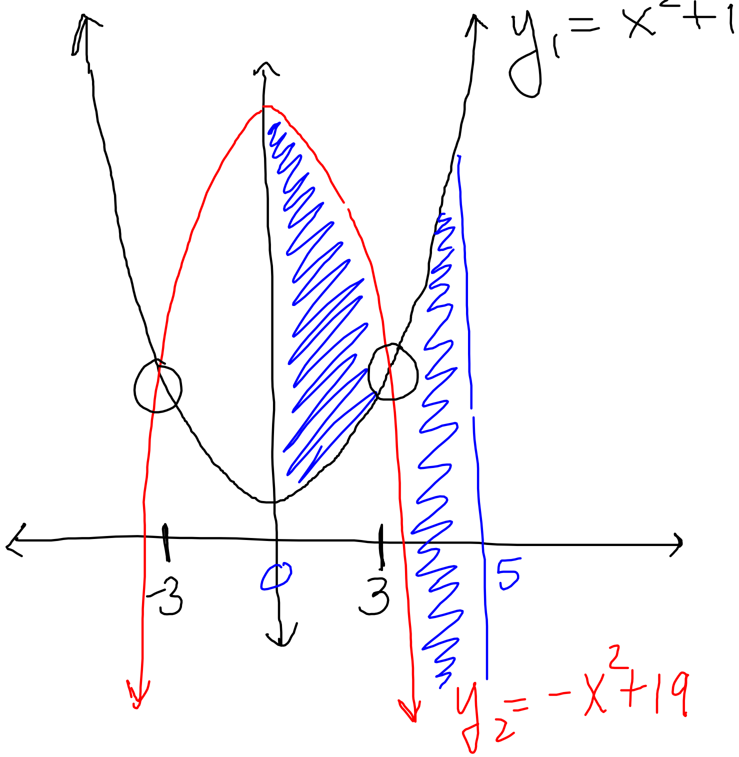
$$\approx \boxed{60.75 \text{ tons.}}$$

14. If the temperature $C(t)$ in an aquarium is made to change according to $C(t) = t^3 - 2t + 10$ for $0 \leq t \leq 2$ (in degrees Celsius), what is the average temperature over the period of time for which the temperature is regulated?

$$\frac{1}{2-0}\int_0^2 (t^3-2t+10)\,dt = \frac{1}{2}\,\text{fnInt}(t^3-2t+10, t, 0, 2)$$

$$= \boxed{10^\circ C}$$

15. Find the area between $y = x^2 + 1$ and $y = -x^2 + 19$ on $\boxed{[0, 5]}$.



$$\text{Area} = \int_0^3 (y_2 - y_1)\,dx + \int_3^5 (y_1 - y_2)\,dx$$

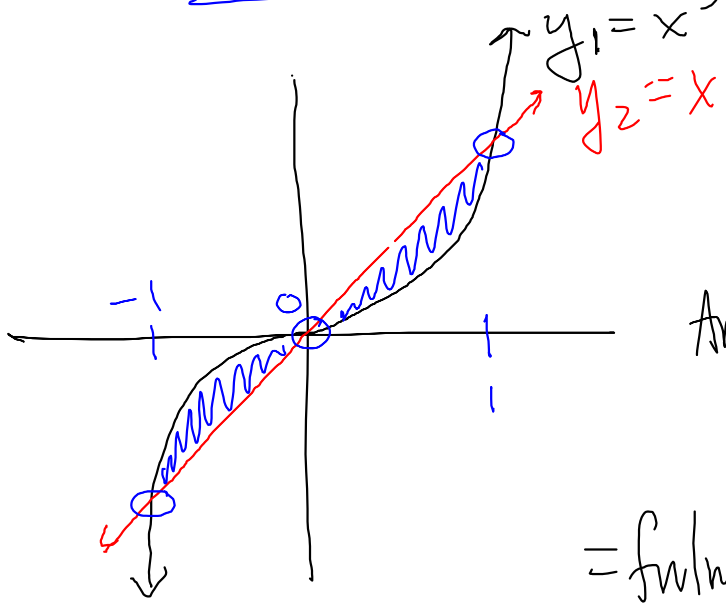
$$= \text{fnInt}(y_2 - y_1, x, 0, 3) + \text{fnInt}(y_1 - y_2, x, 3, 5)$$

$$= \boxed{\frac{196}{3}}$$

Intersection Pt(s)

$$y_1 = y_2$$
$$x^2 + 1 = -x^2 + 19$$
$$2x^2 = 18$$
$$x^2 = 9 \rightarrow x = \pm 3$$

16. Find the area bounded by $y = x^3$ and $y = x$.



$$\text{Area} = \int_{-1}^{0} (y_1 - y_2)\,dx + \int_{0}^{1} (y_2 - y_1)\,dx$$

$$= \text{fnInt}(y_1 - y_2, X, -1, 0) + \text{fnInt}(y_2 - y_1, X, 0, 1)$$

$$= \boxed{1/2}$$

Intersection Pt(s)

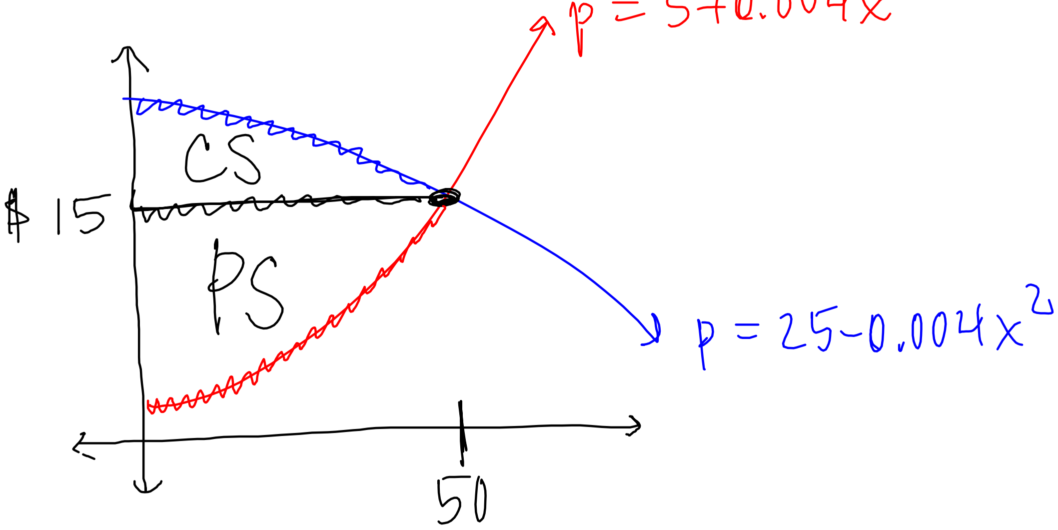
$$x^3 = x$$
$$x^3 - x = 0$$
$$x(x^2 - 1) = 0$$
$$x = 0,\ x = \pm 1$$

17. If supply and demand for a product are given by $p = 5 + 0.004x^2$ and $p = 25 - 0.004x^2$, respectively, find the following at equilibrium price:

Supply demand

(a) consumers' surplus

(b) producers' surplus



EQ PT: $S = D$

$$5 + 0.004x^2 = 25 - 0.004x^2$$

$$0.008x^2 = 20$$

$$x^2 = 2500$$

$$x = \pm 50$$

(a) $CS = \int_0^{50} \left[(25 - 0.004x^2) - 15 \right] dx$

$= \text{fnInt}(25 - 0.004x^2 - 15, x, 0, 50)$

$= \boxed{\$\,333.33}$

(b) $PS = \int_0^{50} \left[15 - (5 + 0.004x^2) \right] dx$

$= \text{fnInt}(15 - 5 - 0.004x^2, x, 0, 50)$

$= \boxed{\$\,333.33}$

$p = 5 + 0.004(50)^2 = 15$

18. The probability that a particular doctor will spend t hours with a patient during an office visit is given by the probability density function $f(t) = \begin{cases} \frac{4}{3}(t+1)^{-2} & , 0 \le t \le 3 \\ 0 & , \text{otherwise} \end{cases}$

(a) What is the probability that this doctor will spend $\underbrace{\text{more than 1 hour}}_{1 \le t \le 3}$ with a randomly selected patient?

(b) What is the probability that this doctor will spend $\underbrace{\text{exactly 1 hour}}_{t=1}$ with a randomly selected patient?

(a) $\int_1^3 \frac{4}{3}(t+1)^{-2}\,dt = \text{fnInt}(4/3\,(t+1)^{-2}, t, 1, 3)$
$\approx \boxed{0.3333}$

(b) $\int_1^1 \frac{4}{3}(t+1)^{-2}\,dt = \boxed{0}$

19. Starting at age 25, you deposit \$1500 a year into a retirement account. If the deposits are treated as a continuous income stream and the money in the account earns 6% compounded continuously,

(a) How much will you have in the account if you retire at age 65?

(b) How much of the final amount is interest?

40 yrs = T

r = 0.06

(a) $FV = e^{0.06(40)} \underbrace{\int_0^{40} 1500 e^{-0.06t}\,dt}_{\text{Fn Int}} \approx \$250{,}579.41$

(b) Deposited = \$1500 × 40

Interest = \$250,579.41 − (1500 × 40)

\$190,579.41