Week-In-Review #9 (6.4, 6.5, 6.6)

1. Given \( f(x) \) is continuous on \((-\infty, \infty)\), write the following as a single integral:

\[
\int_{-2}^{0} f(x) \, dx + \int_{0}^{3} f(x) \, dx - \int_{0}^{9} f(x) \, dx
\]

\[
\int_{-2}^{9} f(x) \, dx = \int_{-2}^{0} f(x) \, dx + \int_{0}^{3} f(x) \, dx - \int_{0}^{9} f(x) \, dx
\]
2. Given $0 < A < B$ and 
\[
\int_0^A x \, dx = \frac{8}{3} \quad \int_0^A x^2 \, dx = \frac{64}{3} \quad \text{and} \quad \int_A^B x^2 \, dx = \frac{152}{3}
\]
the following:

(a) \[\int_0^A 2x^2 \, dx = 2 \int_0^A x^2 \, dx = 2 \left( \frac{64}{3} \right) = \frac{128}{3}\]

(b) \[\int_0^B (-4x^2) \, dx = -4 \int_0^B x^2 \, dx = -4 \left[ \frac{64}{3} + \frac{152}{3} \right] = -288\]

(c) \[\int_0^A (x^2 - 7x) \, dx = \int_0^A x^2 \, dx - \int_0^A 7x \, dx = \frac{64}{3} - 7 \int_0^A x \, dx = \frac{64}{3} - 7(8) = -\frac{104}{3}\]
3. Using the given graph of \( f(x) \), calculate the following:

\[
\int_{a}^{b} f(x) \, dx = \text{"net area between the curve and the x-axis from } x=a \text{ to } x=b \text{.}
\]

(area above x-axis is positive, area below x-axis is negative)

(a) \( \int_{a}^{b} f(x) \, dx = A = \boxed{1.37} \)

(b) \( \int_{b}^{a} f(x) \, dx = -B = \boxed{-2.45} \)

(c) \( \int_{b}^{a} f(x) \, dx = \int_{a}^{b} f(x) \, dx = A - B = 1.37 - 2.45 = \boxed{-1.08} \)

(d) \( \int_{0}^{e} 2f(x) \, dx = 2 \int_{0}^{e} f(x) \, dx = 2 \left[ \int_{0}^{e} f(x) \, dx \right] = 2 \left( A - D + E \right) = 2 \left( 0.84 - 1.44 + 3.2 \right) = \boxed{5.2} \)

(e) \( \int_{c}^{d} f(x) \, dx = -\int_{c}^{d} f(x) \, dx = - \left[ \int_{c}^{d} f(x) \, dx \right] = D = \boxed{1.44} \)

(f) \( \int_{d}^{b} f(x) \, dx = \boxed{0} \)
4. Given \( b \int_{-3}^{5} (x^2 + 2x + 3) \, dx \),

(a) Use the order properties of definite integrals to find bounds for estimating the value of the integral.

\[ m = \text{ABS MIN of } f(x) \text{ on } [a, b] \quad m = \text{ABS MAX of } f(x) \text{ on } [a, b] \]

\[ f(x) \text{ cont on } [-3, 5] \checkmark \]

\[ f'(x) = 2x + 2 \]

CVS: \( f' = 0? \]

\[ 2x + 2 = 0 \]

\[ 2x = -2 \]

\[ x = -1 \quad \text{in interval} \]

\[ f'(x) \text{ does not exist everywhere} \]

\[ 2 \left( \frac{5 - (-3)}{8} \right) \leq \int_{-3}^{5} (x^2 + 2x + 3) \, dx \leq 3 \left( \frac{5 - (-3)}{8} \right) \]

\[ 16 \leq \int_{-3}^{5} (x^2 + 2x + 3) \, dx \leq 304 \]

(b) Find the actual value of the integral and verify your answer is within the bounds found in (a).

\[ \int_{-3}^{5} (x^2 + 2x + 3) \, dx = \frac{x^3}{3} + \frac{x^2}{2} + 3x \]

\[ \left[ \begin{array}{c}
5 \\
-3
\end{array} \right] = \left( \frac{5}{3} + \frac{2}{2} + 3(5) \right) - \left( \frac{(-3)}{3} + (-3)^2 + 3(-3) \right) \]

\[ = \frac{272}{3} \]

Or...

\( f_{n+1} (x^2 + 2x + 3, x, -3, 5) \)
5. Evaluate the following EXACTLY:

(a) \[ \int_{a}^{b} \left( x^3 + \frac{1}{x} \right) \, dx \] (given \( 0 < a < b \))

\[
\left. \frac{x^4}{4} + \ln |x| \right|_{a}^{b} = \left( \frac{b^4}{4} + \ln (b) \right) - \left( \frac{a^4}{4} + \ln (a) \right)
\]

\[
= \frac{1}{4} b^4 + \ln(b) - \frac{1}{4} a^4 - \ln (a)
\]

\[
= \frac{1}{4} \left( b^4 - a^4 \right) + \ln \left( \frac{b}{a} \right)
\]

(b) \[ \int_{2}^{4} (e^t + \pi) \, dt \]

\[
\left. \frac{e^t + \pi t}{1} \right|_{2}^{4} = \left( e^4 + 4\pi \right) - \left( e^2 + 2\pi \right)
\]

\[
= e^4 + 4\pi - e^2 - 2\pi
\]

\[
= \frac{e^4 - e^2 + 2\pi}{1}
\]
\[ x = \int_{1}^{3} \frac{x^2 + 1}{x^3 + 3x} \, dx \]

\[ u = x^3 + 3x \]
\[ du = (3x^2 + 3) \, dx \]
\[ \Rightarrow \frac{1}{3} \, du = \frac{1}{3} \frac{x^2 + 1}{x^3 + 3x} \, dx \]

**Method 1**

\[ \int \frac{1}{3} \frac{1}{u} \, du = \int \frac{1}{3} \ln|u| + C \Rightarrow \frac{1}{3} \ln|x^3 + 3x| + C \]

\[ \Rightarrow \int_{1}^{3} \frac{x^2 + 1}{x^3 + 3x} \, dx = \left. \frac{1}{3} \ln|x^3 + 3x| \right|_{1}^{3} \]
\[ = \left( \frac{1}{3} \ln|3^3 + 3(3)| \right) - \left( \frac{1}{3} \ln|1^3 + 3(1)| \right) \]
\[ = \frac{1}{3} \ln(3^6) - \frac{1}{3} \ln(4) = \frac{1}{3} \ln \left( \frac{3^6}{4} \right) = \frac{1}{3} \ln 9 \]

**Method 2**

\[ x = \int_{1}^{3} \frac{x^2 + 1}{x^3 + 3x} \, dx \]
\[ u = \frac{36}{x} \]
\[ du = \frac{1}{3} \, du \]
\[ \Rightarrow \int_{4}^{9} \frac{1}{3} \ln|u| \, du = \frac{1}{3} \ln(36) - \frac{1}{3} \ln(4) \]

*Def: Int. completely in terms of "u".*
6. Let $F(x)$ be an antiderivative of $f(x)$. If \( \int_{0}^{8} f(x) \, dx = -10 \) and $F(0) = 5$, find $F(8)$.

\[
\int_{0}^{8} f(x) \, dx = F(x) \bigg|_{0}^{8} = F(8) - F(0) = F(8) - 5 \quad \rightarrow \quad F(8) = -10 + 5 \quad \boxed{F(8) = -5}
\]

7. If \( \int_{a}^{b} f(x) \, dx = 32 \) and \( \int_{a}^{b} \left[3f(x) + 5g(x)\right] \, dx = 100 \), find \( \int_{a}^{b} g(x) \, dx \).

\[
\Rightarrow \int_{a}^{b} 5g(x) \, dx = 100 - 3 \int_{a}^{b} f(x) \, dx = 100 - 3 \cdot 32 = 100 - 96 = 4
\]

\[
\Rightarrow 5 \int_{a}^{b} g(x) \, dx = 4 \quad \rightarrow \quad \int_{a}^{b} g(x) \, dx = \frac{4}{5}
\]
8. Use the graph to the right to answer the following.

(a) Write a definite integral(s) to indicate the shaded area in the graph by first defining \( f(x) \) as a piecewise-defined function.

\[
f(x) = \begin{cases} 
6 & , \ x \leq 2 \\
\frac{2x+2}{4-10} & , \ x > 2
\end{cases}
\]

\[
\int_{-3}^{2} 6 \, dx + \int_{2}^{4} \frac{2x+2}{4-2} \, dx
\]

(b) If \( F(x) \) is an antiderivative of \( f(x) \) and \( F(3) = 7 \), find \( F(1) \).

\[
\int_{1}^{3} f(x) \, dx = F(x) \bigg|_{1}^{3} = F(3) - F(1)
\]

\[
2(6) + \frac{1}{2}(1)(2) \quad \Rightarrow \quad 13 = 7 - F(1) \]

\[
F(1) = -6
\]
9. Suppose copper is being extracted from a mine at a rate given by \( y = 100e^{-0.2t} \), where \( t \) is the number of years since mining began and \( y \) is measured in tons of copper/year. At this rate, how much copper (to the nearest ton) will be extracted

(a) during the first 24 months of mining?

\[
\int_0^2 100e^{-0.2t} \, dt = \ln(100e^{-0.2t}) \bigg|_0^2 \\
\approx 165 \text{ tons of Copper}
\]

(b) during the third year of mining?

\[
\int_2^3 100e^{-0.2t} \, dt = \ln(100e^{-0.2t}) \bigg|_2^3 \\
\approx 61 \text{ tons of Copper}
\]
10. If the temperature $C(t)$ in an aquarium is made to change according to $C(t) = t^3 - 2t + 10$ for $0 \leq t \leq 2$ (in degrees Celsius), what is the average temperature over the period of time for which the temperature is regulated?

$$\text{Avg Value of } f(x) \text{ over } [a, b] = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx$$

$$\text{Avg. Temp over } [0, 2] = \frac{1}{2-0} \int_{0}^{2} (t^3 - 2t + 10) \, dt$$

$$= \frac{1}{2} \left[ \frac{1}{4} t^4 - t^2 + 10t \right]_{0}^{2}$$

$$= \frac{1}{2} \left[ \left( \frac{1}{4} \cdot 2^4 - 2^2 + 10 \cdot 2 \right) - \left( \frac{1}{4} \cdot 0^4 - 0^2 + 10 \cdot 0 \right) \right]$$

$$= \frac{1}{2} \left[ 40 \right] = 20 \, ^{\circ}\text{C}$$

$$\boxed{10 \, ^{\circ}\text{C}}$$
11. Find the area between \( y = x^2 + 1 \) and \( y = -x^2 + 19 \),

(a) on the interval \([0, 5]\).

\[
\text{Area} = \int_{0}^{3} \left[ (x^2 + 19) - (x^2 + 1) \right] \, dx
+ \int_{3}^{5} \left[ (x^2 + 1) - (x^2 + 19) \right] \, dx
\]

\[
= \text{fnInt} \left( y_2 - y_1, x, 0, 3 \right) + \text{fnInt} \left( y_1 - y_2, x, 3, 5 \right)
= \frac{196}{3}
\]

(b) on the interval \([-4, 4]\).

\[
\text{Area} = \int_{-4}^{3} \left[ (x^2 + 1) - (x^2 + 19) \right] \, dx
+ \int_{-3}^{3} \left[ (-x^2 + 19) - (x^2 + 1) \right] \, dx
\]

\[
= \text{fnInt} \left( y_1 - y_2, x, -4, -3 \right) + \text{fnInt} \left( y_2 - y_1, x, -3, 3 \right)
+ \text{fnInt} \left( y_1 - y_2, x, 3, 4 \right)
= \frac{256}{3}
\]
12. Find the area bounded by $y = x^3$ and $y = x$.

\[ \text{Area} = \int_{-1}^{0} [x^3 - x] \, dx + \int_{0}^{1} [x - x^3] \, dx \]

\[ = \text{fnInt} (y_2 - y_1, x, -1, 0) + \text{fnInt} (y_1, x, 0, 1) \]

\[ = \frac{1}{2} \]