Theory of PDEs
MATH 412 - 200/501
A. Bonito

December 1
Fall 2016

First Name: ______________________ **Last Name:** ______________________

Midterm 2

- 75 minute individual midterm;
- Answer the questions in the space provided. If you run out of space, continue onto the back of the page. Additional space is provided at the end;
- **Show and explain all work**;
- **Underline** the answer of each steps;
- The use of books, personal notes, **calculator**, cellphone, laptop, and communication with others is forbidden;
- By taking this midterm, you agree to follow the university's code of academic integrity.

Ex 1	Ex 2	Ex 3	Total

Exercise 1 **40%**

1. Compute the cosine transform of e^{-x}

$$\mathcal{C}(e^{-x}) := \frac{2}{\pi}\int_0^\infty e^{-x}\cos(\omega x)\ dx.$$

2. Assume that $\lim_{x\to\infty} f(x) = 0$. Show that

$$C(f') = -\frac{2}{\pi}f(0) + \omega\mathcal{S}(f).$$

3. Assume that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0$. Show that

$$C(f'') = -\frac{2}{\pi}f'(0) - \omega^2 C(f).$$

   Hint: use without proof that $f(x)$ such that $\lim_{x\to\infty} f(x) = 0$, there holds

$$\mathcal{S}(f') = -\omega\mathcal{C}(f).$$

4. Show that the solution of

$$\begin{aligned} \frac{\partial}{\partial t}u - k\frac{\partial^2}{\partial x}u &= 0 \qquad (x > 0) \\ \frac{\partial}{\partial x}u(0,t) &= 0 \\ u(x,0) &= e^{-x} \end{aligned}$$

   is given by

$$u(x,t) = \frac{4}{\pi^2}\int_0^\infty \frac{1}{1+\omega^2}e^{-\omega^2 kt}\cos(\omega x)\ d\omega.$$

Exercise 2 **30%**

Solve

$$\frac{\partial}{\partial t} w + x \frac{\partial}{\partial x} w = 1 \qquad \text{with } w(x,0) = f(x).$$

Find an expression of the form $w(x,t) = ...$

Exercise 3 **30%**

1. Consider the wave equation

$$\frac{\partial^2}{\partial t^2}u - c^2\frac{\partial^2}{\partial x^2}u = 0 \qquad (x \in \mathbb{R})$$

   with initial conditions

$$u(x,0) = \cos(x) \qquad \frac{\partial}{\partial t}u(x,0) = 0.$$

   Take as given that the solution can we written

$$u(x,t) = F(x-ct) + G(x+ct).$$

   Find $F(x)$, $G(x)$ and deduce an expression for $u(x,t)$.

2. Consider now the wave equation on the semi (negative) real line

$$\frac{\partial^2}{\partial t^2}u - c^2\frac{\partial^2}{\partial x^2}u = 0 \qquad (x < 0).$$

   We provide the same initial conditions

$$u(x,0) = \cos(x) \qquad \frac{\partial}{\partial t}u(x,0) = 0.$$

   and set the value at $x = 0$

$$u(0,t) = e^{-t}.$$

   Find an expression for $u(x,t)$.
   Hint: you will have to consider two cases: $x + ct < 0$ and $x + ct > 0$.

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Final 2: solutions

Exercise 1 **40%**

1. We compute

$$\mathcal{C}(e^{-x}) = \frac{2}{\pi}\int_0^\infty e^{-x}\cos(\omega x)\ dx = \frac{2}{\pi}\Re\left(\int_0^\infty e^{-x}e^{i\omega x}dx\right) = \frac{2}{\pi}\Re\left(\int_0^\infty e^{-x(1-i\omega)}dx\right)$$
$$= \frac{2}{\pi}\Re\left(\frac{1}{i\omega-1}(\lim_{x\to\infty} e^{-x(1-i\omega)}-1)\right) = \frac{2}{\pi}\Re\left(\frac{1}{1-i\omega}\right) = \frac{2}{\pi}\Re\left(\frac{1+i\omega}{1+\omega^2}\right) = \frac{2}{\pi}\frac{1}{1+\omega^2}.$$

2. From the definition of the cosine transform we have

$$\mathcal{C}(f') = \frac{2}{\pi}\int_0^\infty f'(x)\cos(\omega x)\ dx = \frac{2}{\pi}\left(\omega\int_0^\infty f(x)\sin(\omega x)\ dx + \lim_{x\to\infty} f(x)\cos(\omega x) - f(0)\right)$$
$$= \omega\mathcal{S}(f) - \frac{2}{\pi}f(0).$$

3. We apply the above relation as we as the hint successively

$$\mathcal{C}(f'') = \omega S(f') - \frac{2}{\pi}f'(0) = -\omega^2 C(f) - \frac{2}{\pi}f'(0).$$

4. We use the cosine transform: $\mathcal{C}(u(.,t)) = U(.,t)$. The boundary condition implies that

$$\mathcal{C}\left(\frac{\partial^2}{\partial x}u(.,t)\right) = -\omega^2 U(.,t)$$

and therefore $U(\omega,t)$ satisfies

$$\frac{\partial}{\partial t}U + \omega^2 kU = 0$$

or

$$U(\omega,t) = B(\omega)e^{-\omega^2 kt}$$

for some function $B(\omega)$ to be determined using the initial condition.
In fact,

$$U(\omega,t) = \frac{2}{\pi}\int_0^\infty u(x,t)\cos(\omega x)\ dx$$

so that

$$B(\omega) = U(\omega,0) = \frac{2}{\pi}\int_0^\infty u(x,0)\cos(\omega x)\ dx = \frac{2}{\pi}\int_0^\infty e^{-x}\cos(\omega x)\ dx = \frac{4}{\pi^2}\frac{1}{1+\omega^2}.$$

Therefore

$$U(\omega,t) = \frac{4}{\pi^2}\frac{1}{1+\omega^2}e^{-\omega^2 kt}$$

and

$$u(x,t) = \frac{4}{\pi^2}\int_0^\infty \frac{1}{1+\omega^2}e^{-\omega^2 kt}\cos(\omega x)\ d\omega.$$

Exercise 2 **30%**

We first determine the characteristics from the ODE

$$\frac{d}{dt}x(t) = x(t), \qquad x(0) = x_0.$$

The solutions are

$$x(t) = x_0 e^t.$$

Hence the transport equations reduces to

$$\frac{d}{dt}w(x_0 e^t, t) = 1, \qquad w(x, 0) = f(x).$$

We integrate the above (exact) ODE from 0 to t to find

$$w(x_0 e^t, t) - w(x_0, 0) = t$$

or

$$w(x_0 e^t, t) = t + f(x_0).$$

Note that $x = x_0 e^t$ and so $x_0 = xe^{-t}$ which leads to the final expression for the solution

$$w(x, t) = t + f(xe^{-t}).$$

Exercise 3 **30%**

1. We use the initial conditions on the superposition formula

$$u(x, t) = F(x - ct) + G(x + ct)$$

to get

$$\cos(x) = F(x) + G(x) \qquad \text{and} \qquad 0 = -cF(x) + cG(x).$$

The solution of this system of two equations / two unknowns is

$$F(x) = G(x) = \frac{1}{2}\cos(x)$$

so that

$$u(x, t) = \frac{1}{2}\cos(x - ct) + \frac{1}{2}\cos(x + ct).$$

2. The above solution remains valid as long as $x + ct < 0$. When $x + ct > 0$, we need to find an expression for $G(s)$ where $s > 0$. To do so, we use the boundary condition $u(0, t) = e^t$ which implies

$$e^{-t} = F(-ct) + G(ct).$$

Setting $s = -ct$, we find

$$G(-s) = e^{s/c} - F(s).$$

Therefore, when $x + ct > 0$

$$G(x + ct) = e^{-(t+x/c)} - F(-x - ct) = e^{-(t+x/c)} - \frac{1}{2}\cos(x + ct).$$

In summary, we find that the solution is given by

$$u(x, t) = \begin{cases} \frac{1}{2}\cos(x - ct) + \frac{1}{2}\cos(x + ct) & \text{when } x + ct < 0 \\ \frac{1}{2}\cos(x - ct) - \frac{1}{2}\cos(x + ct) + e^{-(t+x/c)} & \text{when } x + ct > 0 \end{cases} \qquad \text{and } x < 0.$$