

TWO-LEVEL PRECONDITIONERS FOR 2M'TH ORDER ELLIPTIC FINITE ELEMENT PROBLEMS*

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Abstract. In this paper, we analyze two-level preconditioners for second and fourth order elliptic boundary value problems. These preconditioners involve smoothing on the original problem and the solution (or preconditioning) of an auxiliary problem on a related mesh. Two abstract theorems are provided as a basis for this analysis. Properties needed to apply these theorems are developed for general finite element approximation spaces. These results are then applied to the second order and Biharmonic Dirichlet problems. Uniform preconditioning estimates are proved in the general case when the triangulations are only assumed to be shape regular but not necessarily quasiuniform. In the case when the meshes are of locally comparable size, this analysis applies to both conforming and nonconforming finite element approximations. When the preconditioning mesh is genuinely coarser than the original, an analysis is given in the case that the auxiliary problem is conforming. For this application, it is shown that appropriate smoothers can be defined based on overlapping Schwarz methods.

1. Introduction. In this paper, we consider two-level techniques for constructing preconditioners for the discrete systems which arise from finite element approximation of second and fourth order elliptic boundary value problems. We will consider only finite element approximations to Dirichlet problems with homogeneous boundary conditions. However, our approach can also be used for other types of boundary conditions as well as for the finite difference discretizations.

A two-level preconditioner can be thought of as an additive version of the general two-level multigrid algorithm. Thus, the preconditioner for a problem on a given grid is constructed in terms of a solution or approximate solution of an auxiliary problem on a related grid and a “smoother” on the original grid. Ideally, the related problem should be simpler to solve or precondition. This may be achieved, for example, by using a somewhat coarser grid or an approximation based on a simpler finite element.

Clearly, there are numerous examples of two-level results already in the literature

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since any general nested or nonnested multigrid result implies the corresponding two-level result (see, e.g. [3, 4, 5, 6, 8, 9, 17] and many others). The only reason for considering two level methods on their own is that it is possible to prove results which are stronger than those obtained in the general multilevel setting.

Two level results are easily developed for elements of the same type and slightly refined meshes and reduce to proving the so-called strengthened Cauchy-Schwarz inequality (see, [2, 3, 18, 28]). The case of different elements on the same mesh can often be analyzed by comparing them when mapped to a reference element. Comparisons between conforming and nonconforming elements are then straightforward. Results for mixed and conforming finite element pairs are given in [13] and for mixed and nonconforming finite element pairs are given in [25].

Two level results can also be proved using a so-called “regularity and approximation” approach, which was used in general nonnested multilevel applications [9]. We provide a theorem in this paper which shows that the natural additive two level preconditioner gives rise to uniformly convergent iterations under the regularity and approximation assumption.

Another technique for developing two level estimates is a regularity free approach and can be thought of as a special case of multilevel additive Schwarz methods [8, 14, 15, 22]. The major part of this paper can be thought of as an extension of this approach to nonnested meshes. Other work in this direction has been done in [10, 23, 27]. Although this work extends the two-level case to more general nonnested applications, none of the above references deals with the case of finite element approximations on nonquasiuniform meshes. The theory developed in this paper provides a general approach which is applicable to general mesh triangulations.

In this paper we provide a general approach for analyzing two-level preconditioners for $2m$ -th order elliptic boundary value problems, for $m = 1, 2$. This theory is novel in the use of a weighted norm and a generalized smoothing condition. The weighted norm allows for the application to problems on nonuniform meshes. The generalized smoothing condition allows for the applications to situations where the auxiliary mesh is significantly coarser than the original mesh.

The outline of the remainder of the paper is as follows. In Section 2, we provide abstract theorems for the analysis of two-level preconditioners. The application of this theory requires the verification of stability and approximation results between pairs of finite element spaces. These properties are derived in Section 3. Section 4 applies the general theory to second order boundary value problems. The results for the biharmonic Dirichlet problem are given in Section 5.

2. The two-level algorithm and abstract convergence analysis. In this section, we first define the two-level additive preconditioner. We then consider two sets of hypotheses which imply condition number estimates for the preconditioned system. Theorem 2.1 involves the so-called “regularity and approximation” assumption. This is an old result which has been announced but never published. Theorem 2.2 gives a regularity free result. Although other authors have recently used such an approach, our analysis is novel in that it is general enough to handle applications to nonquasiuniform meshes.

We will discuss the two-level algorithms in an abstract setting. To this end, let V_h and V_H be two finite dimensional spaces equipped with base inner products $(\cdot, \cdot)_h$ and $(\cdot, \cdot)_H$ and *energy* inner products $A_h(\cdot, \cdot)$ and $A_H(\cdot, \cdot)$. We denote the corresponding base norms by $|\cdot|_h$ and $|\cdot|_H$, respectively. Consider the pair of variational problems: Find $u \in V_h$ and $\hat{u} \in V_H$ such that

$$(2.1) \quad A_h(u, \chi) = (f, \chi)_h, \quad \text{for all } \chi \in V_h,$$

$$(2.2) \quad A_H(\hat{u}, \chi) = (f, \chi)_H, \quad \text{for all } \chi \in V_H.$$

Let $A_h : V_h \mapsto V_h$ and $A_H : V_H \mapsto V_H$ be the discrete operators on V_h and V_H defined by

$$(2.3) \quad \begin{aligned} (A_h v, \chi)_h &= A_h(v, \chi), & \text{for all } v, \chi \in V_h, \\ (A_H v, \chi)_H &= A_H(v, \chi), & \text{for all } v, \chi \in V_H. \end{aligned}$$

Our goal is to study preconditioners for A_h using A_H or a good preconditioner for A_H . We will assume that the two spaces V_h and V_H are related by a connection operator $\mathcal{I}_h : V_H \mapsto V_h$. Denote by $\mathcal{I}_h^t : V_h \mapsto V_H$ and $\mathcal{I}_h^* : V_h \mapsto V_H$ the base and energy-adjoints of \mathcal{I}_h defined by

$$(2.4) \quad \begin{aligned} (\mathcal{I}_h^t u, \chi)_H &= (u, \mathcal{I}_h \chi)_h, & \text{for all } u \in V_h, \chi \in V_H, \\ A_H(\mathcal{I}_h^* u, \chi) &= A_h(u, \mathcal{I}_h \chi), & \text{for all } u \in V_h, \chi \in V_H. \end{aligned}$$

Note that $\mathcal{I}_h^* = A_H^{-1} \mathcal{I}_h^t A_h$. Since these operators are adjoints,

$$\max_{\substack{u \in V_H \\ |u|_H=1}} |\mathcal{I}_h u|_h = \max_{\substack{u \in V_h \\ |u|_h=1}} |\mathcal{I}_h^t u|_H \quad \text{and} \quad \max_{\substack{u \in V_h \\ |u|_h=1}} A_H(\mathcal{I}_h^* u, \mathcal{I}_h^* u) = \max_{\substack{u \in V_H \\ A_H(u, u)=1}} A_h(\mathcal{I}_h u, \mathcal{I}_h u).$$

Let $M_h : V_h \mapsto V_h$ be a symmetric, positive definite operator. We will consider the preconditioner for A_h given by

$$(2.5) \quad B_h^{-1} = \mathcal{I}_h A_H^{-1} \mathcal{I}_h^t + M_h^{-1}.$$

Note that

$$T \equiv B_h^{-1}A_h = \mathcal{I}_h\mathcal{I}_h^* + M_h^{-1}A_h.$$

REMARK 2.1. *It is obvious that if B_h results in a good preconditioner for A_h then A_H^{-1} in (2.5) can be replaced by a good preconditioner and the resulting operator is still a good preconditioner for A_h .*

REMARK 2.2. *Let $\{\phi_i\}$ and $\{\Phi_i\}$ denote the nodal bases for V_h and V_H respectively. The operators \mathcal{I}_h must be computable on basis elements of V_H . For the computation of $B_h^{-1}g$, the quantities $(g, \phi_i)_h$, $i = 1, \dots$ are usually given as data. To evaluate $A_H^{-1}\mathcal{I}_h^\dagger g$, one first computes*

$$(\mathcal{I}_h^\dagger g, \Phi_i)_H = (g, \mathcal{I}_h \Phi_i)_h$$

Thus, if the matrix N takes the nodal values of a function $u \in V_H$ to the nodal values for $\mathcal{I}_h u$ then its transpose takes the vector of inner products $\{(g, \phi_i)_h\}$ into $\{(\mathcal{I}_h^\dagger g, \Phi_i)_H\}$. The data $\{(g, \mathcal{I}_h \Phi_i)_h\}$ is exactly that needed to compute $A_H^{-1}\mathcal{I}_h^\dagger g$. The explicit computation of \mathcal{I}_h^\dagger is not required.

The multiplicative version of the above algorithm is a special case of a two-level nonnested multigrid algorithm [7]. The operator M_h plays the role of a smoothing operator in this algorithm while A_H^{-1} is a coarse grid solution operator. Results for the multiplicative algorithm follow from those for the additive and the general theory of [6].

In the first theorem which we shall give for estimating the condition number of the preconditioned system $T = B_h^{-1}A_h$, the regularity and approximation condition is part of the hypothesis. Specifically, we assume that there is an $\alpha \in (0, 1]$ and $\gamma > 0$ such that

$$(2.6) \quad |A_h((I - \mathcal{I}_h\mathcal{I}_h^*)v, v)| \leq \gamma \left(\frac{|A_h v|_h^2}{\lambda} \right)^{2\alpha} A_h(v, v)^{1-\alpha} \quad \text{for all } v \in V_h.$$

Here λ denotes the largest eigenvalue of A_h . When (2.6) holds, it suffices to take M_h to be a simple smoothing operator such as λI .

THEOREM 2.1. *Assume that (2.6) holds and that M_h is given by λI . Then T satisfies the inequalities*

$$\min(\gamma^{-1/\alpha}, \alpha) A_h(v, v) \leq A_h(Tv, v) \leq (2 + \gamma)A_h(v, v) \quad \text{for all } v \in V_h.$$

Proof. By a generalized arithmetic-geometric mean inequality and (2.6),

$$|A_h((I - \mathcal{I}_h \mathcal{I}_h^*)v, v)| \leq \gamma \left(\alpha \beta^{1-1/\alpha} \frac{|A_h v|_h^2}{\lambda} + (1 - \alpha) \beta A_h(v, v) \right)$$

holds for any positive β and any v in V_h . Taking $\beta = 1$, the upper inequality of the theorem then follows from

$$\begin{aligned} A_h(Tv, v) &= A_h((\mathcal{I}_h \mathcal{I}_h^* - I)v, v) + A_h(v, v) + \lambda^{-1} |A_h v|_h^2 \\ &\leq (\gamma + 2) A_h(v, v). \end{aligned}$$

For the lower inequality, we take $\beta = 1/\gamma$ and get

$$\begin{aligned} A_h(v, v) &\leq A_h((I - \mathcal{I}_h \mathcal{I}_h^*)v, v) + A_h(\mathcal{I}_h \mathcal{I}_h^* v, v) \\ &\leq \alpha \gamma^{1/\alpha} \lambda^{-1} |A_h v|_h^2 + (1 - \alpha) A_h(v, v) + A_h(\mathcal{I}_h \mathcal{I}_h^* v, v). \end{aligned}$$

The lower inequality follows from obvious manipulations. This completes the proof of the theorem. \square

REMARK 2.3. *The technique of using elliptic regularity to analyze two-level additive preconditioners is not new. In fact, such results were announced by J. Bramble at a Finite Element Circus at Cornell University in the mid eighties.*

There are many applications where regularity and approximation estimates of the form of (2.6) are known. However, there are applications where such estimates are not available. In particular, regularity and approximation estimates with uniform γ are not known in the case of elliptic finite element problems on meshes which are not quasiuniform. To handle this situation, we provide a second theorem for the additive preconditioner. For this theorem we require that the spaces V_h and V_H belong to a common (possibly infinite dimensional) normed linear space \mathcal{V} . The norm on \mathcal{V} will be denoted $\|\cdot\|_\Lambda$. As hypotheses for the theorem, we introduce the following conditions.

(a) The operator M_h behaves like a smoother; i.e.,

$$\eta_1^{-1} A_h(u, u) \leq (M_h u, u)_h \leq \eta_2 [\|u\|_\Lambda^2 + A_h(u, u)], \quad \text{for all } u \in V_h.$$

(b) V_H approximates V_h in the sense that there exists a positive number α such that

$$\inf_{\chi \in V_H} \{ \|u - \chi\|_\Lambda^2 + A_H(\chi, \chi) \} \leq \alpha A_h(u, u), \quad \text{for all } u \in V_h.$$

(c) The operator \mathcal{I}_h provides stable approximation to V_H in the sense that there exists a positive number β such that

$$\|\mathcal{I}_h u - u\|_\Lambda^2 + A_h(\mathcal{I}_h u, \mathcal{I}_h) \leq \beta A_H(u, u), \quad \text{for all } u \in V_H.$$

The following theorem shows that the above conditions are sufficient to provide an estimate for the preconditioned system T .

THEOREM 2.2. *Assume that conditions (a)–(c) hold for \mathcal{I}_h , V_h and V_H . Then*

$$C_1 A_h(u, u) \leq A_h(Tu, u) \leq C_2 A_h(u, u) \quad \text{for all } u \in V_h,$$

or equivalently

$$C_1 (B_h u, u)_h \leq (A_h u, u)_h \leq C_2 (B_h u, u)_h \quad \text{for all } u \in V_h.$$

Here $C_1 = \alpha + 2\eta_2[1 + \alpha(1 + \beta)]$ and $C_2 = \beta + \eta_1$.

Proof. Since $T = \mathcal{I}_h \mathcal{I}_h^* + M_h^{-1} A_h$, we have that for any $u \in V_h$,

$$\begin{aligned} A_h(Tu, u) &= A_h(\mathcal{I}_h \mathcal{I}_h^* u, u) + A_h(M_h^{-1} A_h u, u) \\ &\leq (A_H(\mathcal{I}_h^* u, \mathcal{I}_h^* u) + \eta_1 A_h(u, u)) \leq (\beta + \eta_1) A_h(u, u). \end{aligned}$$

This proves the upper bound of the theorem.

Let u be in V_h and $\chi \in V_H$ satisfy

$$(2.7) \quad \|u - \chi\|_\Lambda^2 + A_H(\chi, \chi) \leq \alpha A_h(u, u).$$

For convenience, we denote $w = u - \mathcal{I}_h \chi$. Using the Cauchy-Schwarz inequality and the definition of B_h^{-1} , we obtain

$$\begin{aligned} (B_h u, u)_h &= (B_h u, \mathcal{I}_h \chi)_h + (B_h u, w)_h = (\mathcal{I}_h^t B_h u, \chi)_H + (B_h u, w)_h \\ &\leq \left((A_H^{-1} \mathcal{I}_h^t B_h u, \mathcal{I}_h^t B_h u)_H + (M_h^{-1} B_h u, B_h u)_h \right)^{1/2} \\ &\quad \cdot \left((A_H \chi, \chi)_H + (M_h w, w)_h \right)^{1/2} \\ &= (B_h u, u)_h^{1/2} \left((A_H \chi, \chi)_H + (M_h w, w)_h \right)^{1/2}. \end{aligned}$$

This proves that

$$(2.8) \quad \begin{aligned} (B_h u, u)_h &\leq (A_H \chi, \chi)_H + (M_h(u - \mathcal{I}_h \chi), u - \mathcal{I}_h \chi)_h \\ &\leq \alpha A_h(u, u) + (M_h(u - \mathcal{I}_h \chi), u - \mathcal{I}_h \chi)_h. \end{aligned}$$

By Conditions (a), (c) and (2.7),

$$\begin{aligned} (M_h(u - \mathcal{I}_h \chi), u - \mathcal{I}_h \chi)_h &\leq \eta_2 [\|u - \mathcal{I}_h \chi\|_\Lambda^2 + A_h(u - \mathcal{I}_h \chi, u - \mathcal{I}_h \chi)] \\ &\leq 2\eta_2 [\|u - \chi\|_\Lambda^2 + \|\chi - \mathcal{I}_h \chi\|_\Lambda^2 + A_h(u, u) + A_h(\mathcal{I}_h \chi, \mathcal{I}_h \chi)] \\ &\leq 2\eta_2 [1 + \alpha(1 + \beta)] A_h(u, u). \end{aligned}$$

Combining the above estimates completes the proof of the theorem. \square

REMARK 2.4. *The basic approach for proving the above theorem is standard. Variations of this technique has appeared in [8, 14, 15, 20, 24], etc. In addition, other two-level theorems are now appearing [10, 23, 27]. The results in [10] and [27] can only be applied to finite element applications with quasiuniform spaces. Our theorem is interesting because it provides explicit estimates in a framework which allows for the application to meshes which are not quasiuniform. Oswald's results [23] are abstract and could be combined with our hypotheses to provide a result under the additional assumption that $(M_h v, v)$ and $\|v\|_{\Lambda}^2$ are equivalent for all functions v in V_h . This stronger condition on the smoother is not satisfied in one of our applications.*

3. Some stability estimates for finite element spaces. In this section, we provide simultaneous approximation results for pairs of nonconforming nodal finite element approximation spaces. Since a conforming finite element space is a special case, the results also apply when one or both of the spaces are conforming. We first define the approximation spaces. Next, we prove a Bramble-Hilbert Lemma for the nonconforming space. Finally, we prove the simultaneous approximation results.

We start by breaking up the domain Ω into triangles $\mathcal{T}_h = \{\tau\}$ in the usual way. The triangles are closed with non-intersecting interiors. Here $h = \max_{\tau \in \mathcal{T}_h} h_{\tau}$ where $h_{\tau} = \text{diam}(\tau)$. Throughout this paper, the partitioning is assumed to be non-degenerate so that the intersection of two triangles is a point or an edge of the triangles. In addition, the partitioning is assumed to satisfy a minimum angle condition, i.e., each angle in the triangulation is greater than or equal to some fixed angle θ_0 . This implies that the diameters of neighboring triangles are roughly the same and only a fixed number (independent of h) of triangles can meet at any point. To avoid technical anomalies, we assume that the mesh can be extended to a shape regular mesh on \mathbf{R}^2 .

The techniques to be developed extend without difficulty to some other cases, e.g. quadrilaterals, as well as higher dimensional problems. We restrict our attention to the case of triangles in \mathbf{R}^2 for simplicity of presentation.

We consider families of finite element spaces indexed by $h \in (0, 1]$. The finite element space V_h is defined to be the span of nodal basis functions $\{\phi_i, i = 1, \dots, n\}$. Associated with each nodal basis function, there is a node x_i and a differential operator \mathcal{L}_i , all terms of which are of degree $\alpha_i \leq m$. The supports of the function ϕ_i is contained in the union of the triangles containing x_i . For sufficiently smooth v , the

nodal interpolation operator Π_h is defined by

$$\Pi_h v = \sum_{i=1}^n (\mathcal{L}_i v)(x_i) \phi_i.$$

We shall assume that, by construction, $\Pi_h v = v$ for all $v \in V_h$. To impose the boundary condition, we exclude nodes on the boundary of degree less than m .

REMARK 3.1. *The coefficients defining \mathcal{L}_i are of unit size. Thus, \mathcal{L}_i could be a point value, an x or y derivative, a normal derivative, etc.*

REMARK 3.2. *The nodes on the edges of the triangles are continuity nodes since they impose some interelement continuity constraints on the approximation subspace. Conforming spaces have edge nodes which result in spaces of functions which are in C^{m-1} . It follows that functions in conforming spaces and their derivatives of order less than m vanish on the boundary.*

Let $W^{s,p}(\Omega)$ and $H^s(\Omega) = W^{s,2}(\Omega)$ denote the usual Sobolev spaces on Ω [16, 19, 21]. Denote by $\|\cdot\|_{s,p,D}$ and $\|\cdot\|_{s,D} \equiv \|\cdot\|_{s,2,D}$ the $W^{s,p}(D)$ and $H^s(D)$ norms over $D \subset \Omega$, and denote by $|\cdot|_{s,p,D}$ and $|\cdot|_{s,D}$ the respective semi-norms involving only the highest derivatives. On spaces of nonconforming finite elements, we will use additional mesh dependent norms. For any subdomain $\tilde{\tau} \subseteq \Omega$, we define the semi-norm

$$|u|_{s,\tilde{\tau};\mathcal{T}_h}^2 = \sum_{\tau \in \mathcal{T}_h} |u|_{s,\tau \cap \tilde{\tau}}^2.$$

The corresponding full norm is give by

$$\|u\|_{s,\tilde{\tau};\mathcal{T}_h}^2 = \sum_{i=1}^s |u|_{i,\tilde{\tau};\mathcal{T}_h}^2.$$

The nodes in finite element examples appear symmetrically at the vertices, center of edges, or barycenter of the triangle. Furthermore, restriction of any finite element space to a triangle results in a space V_τ which only depends on the location of the vertices of the triangle τ . Let τ_1 and τ_2 be two triangles satisfying the angle condition with diameters in the interval $[1/2, 1]$. Consider two functions

$$w_1 = \sum_i c_i \phi_i^{\tau_1} \quad \text{and} \quad w_2 = \sum_i c_i \phi_i^{\tau_2}$$

where $\phi_i^{\tau_j}$ are the nodal basis functions with respect to the triangle τ_j for $j = 1, 2$. Let $v^j = (v_1^j, v_2^j, v_3^j)$ be the vertices of τ_j considered as an element of \mathbf{R}^6 . We assume that

$$(3.1) \quad C_0 \|w_j\|_{s,\tau_j}^2 \leq \sum_i c_i^2 \leq C_1 \|w_j\|_{s,\tau_j}^2 \quad \text{for } s = 0, \dots, m,$$

$$(3.2) \quad \left| \int_{\tau_1} D^\beta w_1 dx - \int_{\tau_2} D^\beta w_2 dx \right| \leq C |v^1 - v^2| \left(\sum_i c_i^2 \right)^{1/2} \text{ for } |\beta| < m,$$

$$(3.3) \quad \left| |w_1|_{s, \tau_j}^2 - |w_2|_{s, \tau_2}^2 \right| \leq C |v^1 - v^2| \sum_i c_i^2 \text{ for } s = 0, \dots, m.$$

In (3.2), β is a multi-index and D^β is the differential operator

$$D^\beta = \left(\frac{\partial}{\partial x_1} \right)^{\beta^1} \left(\frac{\partial}{\partial x_2} \right)^{\beta^2}.$$

As usual, $|\beta| = \beta^1 + \beta^2$ denotes the order of the differential operator. Here and in the remainder of this paper, C with or without subscript represents a generic positive constant which is independent of the mesh parameters. These constants take on different values in different occurrences. The above assumptions can be verified for the finite element examples which will be considered later.

We shall make the following additional assumptions concerning the finite element spaces.

ASSUMPTION 3.1. *Let $\tau \in \mathcal{T}_h$ be such that $\tau \cap \partial\Omega = \emptyset$. Then for every $p \in \mathcal{P}_{m-1}$, $\Pi_h p = p$ on τ .*

ASSUMPTION 3.2. *There exists a constant C not depending on h such that for all $v \in V_h$ and $\tau \in \mathcal{T}_h$,*

$$(3.4) \quad |v|_{s, q, \tau} \leq C h_\tau^{2(\frac{1}{q} - \frac{1}{p})} |v|_{s, p, \tau}, \quad 1 \leq p, q \leq \infty, 0 \leq s \leq m.$$

ASSUMPTION 3.3. *There exists a constant C not depending on h such that for each $\tau \in \mathcal{T}_h$ and any nodal basis function ϕ_i ,*

$$(3.5) \quad |\phi_i|_{s, q, \tau} \leq C h_\tau^{2/q + \alpha - s}, \quad 1 \leq q \leq \infty, 0 \leq s \leq m.$$

Here α is the order of the derivative associated with ϕ_i .

ASSUMPTION 3.4. *A polynomial $p \in \mathcal{P}_{m-1}$ is completely determined by its nodal values on the nodes of degree less than m on any edge of the triangulation. In particular, if $p \in \mathcal{P}_{m-1}$ vanishes on these nodes on any edge of a triangle then $p \equiv 0$.*

These assumptions hold for the standard finite element spaces used in applications. This will be discussed further in the remaining sections. Assumption 3.1 is generally too weak to provide useful convergence estimates however it suffices for the the analysis of the preconditioning properties of two-level methods considered in the subsequent sections.

A mesh subdomain is a union of triangles in \mathcal{T}_h . A strongly connected mesh subdomain $\tilde{\tau}$ is a connected mesh subdomain such that every two points in $\tilde{\tau}$ can be

connected by a path which crosses the boundaries of the triangles through the edges (and not the vertices). If $v \in V_h$ satisfies

$$(3.6) \quad |v|_{m, \tilde{\tau}; \mathcal{T}_h} = 0$$

then v is a piecewise polynomial of degree less than m with respect to the triangles of $\tilde{\tau}$. If $\tilde{\tau}$ is strongly connected then Assumption 3.4 implies that v is the same polynomial on $\tilde{\tau}$.

Let τ be a triangle in \mathcal{T}_h and define

$$N_1(\tau : \mathcal{T}_h) = \cup_{\hat{\tau} \in \mathcal{T}_h, \hat{\tau} \cap \tau \neq \emptyset} \hat{\tau}$$

and for $l = 2, \dots$ define

$$N_l(\tau : \mathcal{T}_h) = \cup_{\hat{\tau} \in N_{l-1}(\tau : \mathcal{T}_h)} N_1(\hat{\tau} : \mathcal{T}_h).$$

We now state and prove a variant of the Bramble–Hilbert lemma for nonconforming elements.

LEMMA 3.1. *Fix $l \geq 1$. Let τ be in \mathcal{T}_h and set $\tilde{\tau} = N_l(\tau : \mathcal{T}_h)$. If $\tilde{\tau} \cap \partial\Omega = \emptyset$ and $u \in V_h$ satisfies*

$$(3.7) \quad \sum_{\hat{\tau} \in \tilde{\tau}} \int_{\hat{\tau}} D^\beta u \, dx = 0 \quad \text{for all } |\beta| < m$$

then for $s = 0, \dots, m$,

$$(3.8) \quad |u|_{s, \tilde{\tau}; \mathcal{T}_h} \leq C(l) h_\tau^{m-s} |u|_{m, \tilde{\tau}; \mathcal{T}_h}.$$

The constant $C(l)$ only depends on l and θ_0 . In addition, if $\tilde{\tau}$ contains an edge of $\partial\Omega$ then (3.8) holds for all $u \in V_h$.

Proof. For simplicity, we prove the result for $l = 1$. The proof for $l > 1$ is the same. We consider all possible triangulations of \mathbf{R}^2 satisfying the minimal angle condition. We first provide a sketch of the proof and then give the proof in detail.

Our goal is to show that for triangles $\{\tau\}$ with unit diagonal,

$$(3.9) \quad \|u\|_{s, \tilde{\tau}; \mathcal{T}_h} \leq C(l) |u|_{m, \tilde{\tau}; \mathcal{T}_h}.$$

The result of the theorem follows from (3.9) and a standard dilation argument. The set of all possible mesh patches can be partitioned into configurations $\{\mathcal{S}\}$ according to the number of triangles meeting at each vertex of τ (of unit diameter). Each mesh patch in a given configuration can be considered as a point $X(N_1(\tau : \mathcal{T}_h))$ in $\mathbf{R}^{2\tilde{n}}$.

The point is defined by the coordinate values of the vertices of the mesh patch. Here \tilde{n} is the number of vertices in every mesh patch associated with the configuration. Without loss of generality, we may fix one of the vertices to be the origin. Then the set of such points corresponding to a configuration is compact because of the angle condition and the fact that the diameter of τ is one. We consider the mapping $\lambda : \mathcal{S} \mapsto \mathbf{R}$ defined by

$$\lambda(X(N_1(\tau : \mathcal{T}_h))) = \min_{u \neq 0} \frac{|u|_{m, \tilde{\tau}; \mathcal{T}_{\tilde{\tau}}}^2}{\|u\|_{m, \tilde{\tau}; \mathcal{T}_{\tilde{\tau}}}^2}.$$

The functions u in the above minimization satisfy integral constraints of the form of (3.7) when dealing with the ‘‘interior’’ case. We show that λ is a continuous function and hence has a minimum on the set of points corresponding to a configuration. We then argue that $\lambda(X(N_1(\tau : \mathcal{T}_h)))$ is nonzero for any $N_1(\tau : \mathcal{T}_h)$ and hence there is a positive minimum value valid over all points in the configuration. We denote this minimum by $\lambda_{\mathcal{S}}$. There is only a finite number of configurations and hence

$$C(1) \leq \max_{\mathcal{S}} \lambda_{\mathcal{S}}^{-1} < \infty,$$

i.e., (3.9) holds for $l = 1$.

We now provide the full details of the proof just described. Let τ be a triangle in a triangulation of \mathbf{R}^2 with diameter equal to one and set $\tilde{\tau} = N_1(\tau : \mathcal{T}_h)$. Label the nodes of τ , v_1, v_2, v_3 and let $n_i(\tau)$ be the number of triangles of $\tilde{\tau}$ meeting at v_i . The set of all such $\tilde{\tau}$ can be partitioned into configurations according to the values of $n_i(\tau)$, $i = 1, 2, 3$. Because of the minimal angle condition, $n_i(\tau) \leq \bar{n}$ for some number \bar{n} independent of h . Consequently, there are at most \bar{n}^3 configurations.

Consider a fixed configuration corresponding to (n_1, n_2, n_3) . Clearly we can label all of the nodes $v_1, \dots, v_{\tilde{n}}$ in any $\tilde{\tau}$ belonging to this configuration in a unique way. Here $\tilde{n} = n_1 + n_2 + n_3 - 6$ is the number of vertices of $\tilde{\tau}$. Figure 1 illustrates one possible labeling which numbers the outer nodes in a clockwise direction, starting at the node which shares a triangle with vertices v_1 and v_2 of the original triangle. Thus we can think of any $\tilde{\tau}$ as a point $X(\tilde{\tau})$ in $\mathbf{R}^{2\tilde{n}}$. Let

$$(3.10) \quad \mathcal{S} = \{X(\tilde{\tau}) \mid (n_1(\tau), n_2(\tau), n_3(\tau)) = (n_1, n_2, n_3) \text{ and } \text{diam}(\tau) = 1\}.$$

Clearly, \mathcal{S} is a compact subset of $\mathbf{R}^{2\tilde{n}}$.

For each point $X(\tilde{\tau}) \in \mathcal{S}$ let

$$\mathcal{W}_{\tilde{\tau}} = \{u \in V_{\tilde{\tau}} \mid \sum_{\hat{\tau} \in \tilde{\tau}} \int_{\hat{\tau}} D^{\beta} u \, dx = 0 \text{ for } |\beta| < m\}.$$

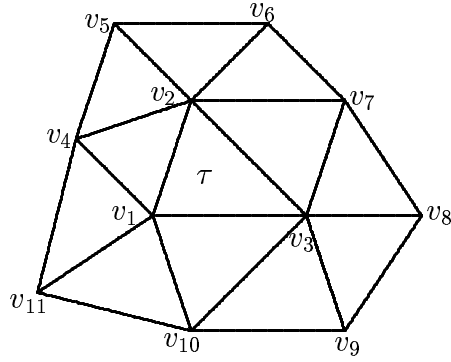


Figure 1.

The node numbering of a typical configuration.

Here $V_{\tilde{\tau}}$ is the nonconforming finite element space associated with the triangulation $\tilde{\tau}$. This space has the same continuity constraints as the original construction of V_h but does not satisfy boundary conditions.

We now show that the map $\lambda : \mathcal{S} \mapsto \mathbf{R}$ defined by

$$\lambda(X(\tilde{\tau})) = \min_{\substack{u \in \mathcal{W}_{\tilde{\tau}} \\ u \neq 0}} \frac{|u|_{m, \tilde{\tau}; \mathcal{T}_{\tilde{\tau}}}^2}{\|u\|_{m, \tilde{\tau}; \mathcal{T}_{\tilde{\tau}}}^2}$$

is continuous. Here $\mathcal{T}_{\tilde{\tau}}$ is the collection of triangles associated with $\tilde{\tau}$. Let $X(\tilde{\tau}_1)$ and $X(\tilde{\tau}_2)$ be two points of \mathcal{S} . Let $w_1 \in \mathcal{W}_{\tilde{\tau}_1}$ be a minimizer of the quotient $|u|_{m, \tilde{\tau}_1; \mathcal{T}_{\tilde{\tau}_1}}^2 / \|u\|_{m, \tilde{\tau}_1; \mathcal{T}_{\tilde{\tau}_1}}^2$ normalized so that

$$\|w_1\|_{m, \tilde{\tau}_1; \mathcal{T}_{\tilde{\tau}_1}} = 1.$$

We can write

$$w_1 = \sum c_i \phi_i^{\tilde{\tau}_1}$$

where the sum is taken over all nodes of $\tilde{\tau}_1$ and $\{\phi_i^{\tilde{\tau}_1}\}$ is the set of shape functions associated with the triangulation $\tilde{\tau}_1$. Define

$$w_2 = \sum c_i \phi_i^{\tilde{\tau}_2} - p$$

where p is the unique polynomial in \mathcal{P}_{m-1} such that

$$\sum_{\tilde{\tau} \in \tilde{\tau}_2} \int_{\tilde{\tau}} D^\beta w_2 \, dx = 0 \text{ for } |\beta| < m.$$

It follows from (3.1), (3.2) and the assumptions on $\tilde{\tau}_2$ that

$$(3.11) \quad \|p\|_{m, \tilde{\tau}_2} \leq C \left| \sum_{|\beta| < m} \int_{\tilde{\tau}_2} D^\beta p \, dx \right| \leq C |X(\tilde{\tau}_1) - X(\tilde{\tau}_2)|.$$

Given ϵ , it easily follows from (3.11) and (3.3) that there exists a δ such that

$$(3.12) \quad |X(\tilde{\tau}_1) - X(\tilde{\tau}_2)| < \delta$$

implies

$$(3.13) \quad \frac{|w_1|_{m, \tilde{\tau}_1: \mathcal{T}_{\tilde{\tau}_1}}^2}{\|w_1\|_{m, \tilde{\tau}_1: \mathcal{T}_{\tilde{\tau}_1}}^2} \geq \frac{|w_2|_{m, \tilde{\tau}_2: \mathcal{T}_{\tilde{\tau}_2}}^2}{\|w_2\|_{m, \tilde{\tau}_2: \mathcal{T}_{\tilde{\tau}_2}}^2} - \epsilon.$$

Since $\lambda(X(\tilde{\tau}_2))$ minimizes the Rayleigh quotient, (3.13) implies that

$$\lambda(X(\tilde{\tau}_2)) \leq \lambda(X(\tilde{\tau}_1)) + \epsilon$$

when (3.12) holds. A similar argument shows that (3.12) implies the opposite inequality

$$\lambda(X(\tilde{\tau}_1)) \leq \lambda(X(\tilde{\tau}_2)) + \epsilon.$$

Thus, the map $X(\tilde{\tau}) \mapsto \lambda(X(\tilde{\tau}))$ is continuous.

It follows that there exists a patch $\tilde{\tau}'$ with $X(\tilde{\tau}') \in \mathcal{S}$ for which $\lambda(X(\tilde{\tau}'))$ is a minimum. The mesh subdomain $\tilde{\tau}'$ is strongly connected and following the discussion after (3.6), $|\cdot|_{m, \tilde{\tau}': \mathcal{T}'}$ provides a norm on $\mathcal{W}(\tau')$ and hence this minimum is nonzero. If we then take λ to be the minimum of these values over all of the configurations (satisfying the minimal angle condition) we get that for each τ with $\text{diam}(\tau) = 1$,

$$(3.14) \quad \|u\|_{m, \tilde{\tau}: \mathcal{T}_{\tilde{\tau}}} \leq \lambda^{-1} |u|_{m, \tilde{\tau}: \mathcal{T}_{\tilde{\tau}}}$$

for all $u \in V_h$ satisfying (3.7). The lemma in the case of $l = 1$ and $\tilde{\tau} \cap \partial\Omega = \emptyset$ follows by a standard dilation argument mapping an arbitrary mesh triangle τ to one with unit diameter.

We handle the case in which $\tilde{\tau}$ shares an edge with the boundary as follows. Since the mesh triangulation can be extended to \mathbf{R}^2 , any $\tilde{\tau}$ is a mesh subdomain of a patch $\bar{\tau}$ in \mathbf{R}^2 corresponding to one of the configurations enumerated above. Note that $\bar{\tau}$ may be equal to $\tilde{\tau}$ or contain additional triangles not inside Ω . Let v be in $V_{\tilde{\tau}}$ and \bar{v} denote the extension of v defined by assigning the value zero to nodes not in $\bar{\Omega}$. Thus it suffices to prove (3.14) for functions in $V_{\bar{\tau}}$ which vanish on the nodes (of degree less than m) of at least one edge, say Γ . Each configuration used above produces a number of new configurations, one for each edge on the boundary. The spaces corresponding to the new configurations have zero nodal values for the nodes of degree less than m on Γ .

We repeat the argument given above for an interior $\tilde{\tau}$ replacing $\tilde{\tau}$ with $\bar{\tau}$ and define $\mathcal{W}_{\bar{\tau}}$ to be the set of functions in $V_{\bar{\tau}}$ which vanish on the nodes (of degree less than m) of Γ . As above, for any configuration, there is a $\bar{\tau}'$ which minimizes the quantity $\lambda(X(\bar{\tau}))$ over all $\bar{\tau}$ in the configuration. This minimum is nonzero because of Assumption 3.4. The inequality (3.14) follows for all $\bar{\tau}$ in any of the new configurations. This completes the proof of the lemma \square

REMARK 3.3. *The above lemma does not provide a result for a patch $\tilde{\tau} = N_l(\tau : \mathcal{T}_h)$ which only intersects $\partial\Omega$ at a vertex.*

Now let \mathcal{T}_H be another triangulation of Ω and V_H be a conforming or nonconforming finite element space defined with respect to \mathcal{T}_H . The meshes \mathcal{T}_h and \mathcal{T}_H need not be related and the subspaces V_h and V_H may consist of elements of different types.

We will define a grid transfer operator $\mathcal{I}_h : V_H \mapsto V_h$. Let \mathcal{L}_i , x_i and ϕ_i denote respectively the differential operators, the nodal values and basis functions corresponding to the subspace V_h . Fix $v \in V_H$ and let x_i be a node of \mathcal{T}_h with $\alpha_i < m$. The quantity $\mathcal{L}_i v(x_i)$ has possibly multiple values when x_i is a node on an edge of the triangulation \mathcal{T}_H . If $\mathcal{L}_i v(x_i)$ is single valued, we set $F_i(v) = \mathcal{L}_i v(x_i)$. Otherwise, we take $F_i(v)$ to be an average of the values of $\mathcal{L}_i v(x_i)$ from the neighboring triangles. We then define

$$(3.15) \quad \mathcal{I}_h v = \sum_{\alpha_i < m} F_i(v) \phi_i.$$

The next result shows that \mathcal{I}_h satisfies a stable approximation property between pairs of conforming or nonconforming finite element spaces. These results are well known in the case of conforming finite element approximation to second order problems on quasiuniform meshes.

THEOREM 3.2. *Assume that the meshes are comparable so that for any $\tau_H \in \mathcal{T}_H$ and $\tau_h \in \mathcal{T}_h$ with $\tau_H \cap \tau_h \neq \emptyset$,*

$$(3.16) \quad C_0 h_{\tau_H} \leq h_{\tau_h} \leq C_1 h_{\tau_H}.$$

Let \mathcal{I}_h be defined by (3.15). Then there exists a constant C such that

$$(3.17) \quad \sum_{\tau \in \mathcal{T}_h} h_{\tau}^{-2m} |\mathcal{I}_h u - u|_{0,\tau}^2 + |\mathcal{I}_h u|_{m,\Omega;\mathcal{T}_h}^2 \leq C |u|_{m,\Omega;\mathcal{T}_H}^2 \quad \text{for all } u \in V_H$$

Proof. Let τ be a triangle of \mathcal{T}_h and $\tau_H(\tau)$ be a triangle of \mathcal{T}_H which intersects τ . Because of (3.16) and the angle condition, there exists an integer l which can be chosen independently of \mathcal{T}_h and \mathcal{T}_H such that

$$\tau \subseteq N_l(\tau_H(\tau) : \mathcal{T}_H).$$

We assign $N_l(\tau_H(\tau) : \mathcal{T}_H)$ to τ if either $N_l(\tau_H(\tau) : \mathcal{T}_H) \cap \partial\Omega = \emptyset$ or $N_l(\tau_H(\tau) : \mathcal{T}_H)$ contains an edge of $\partial\Omega$. Alternatively, we assign $N_{l+1}(\tau_H(\tau) : \mathcal{T}_H)$ to τ . Note that if $N_l(\tau_H(\tau) : \mathcal{T}_H)$ intersects $\partial\Omega$ only at vertices of the triangulation \mathcal{T}_H then $N_{l+1}(\tau_H(\tau) : \mathcal{T}_H)$ contains an edge of $\partial\Omega$. Lemma 3.1 can be applied to each of the assigned neighborhoods. Moreover, each point of Ω appears in at most a fixed number (independent of \mathcal{T}_h and \mathcal{T}_H) of such neighborhoods.

Fix $u \in V_H$ and suppose τ is a triangle of \mathcal{T}_h whose assigned neighborhood $\tilde{N} = N_l(\tau_H(\tau) : \mathcal{T}_H)$ does not intersect $\partial\Omega$. If x_i is a node (with $\alpha_i < m$) in τ and p is in \mathcal{P}_{m-1} then

$$F_i(p) = \mathcal{L}_i p(x_i)$$

and hence by Assumption 3.1, for any $p \in \mathcal{P}_{m-1}$,

$$\mathcal{I}_h(p) = p \quad \text{on } \tau.$$

Choose p such that

$$\int_{\tau} D^{\beta}(u - p) \, dx = 0 \quad \text{for all } |\beta| < m.$$

Set $\hat{u} = u - p$. We clearly have that for $s = 0$ or $s = m$,

$$|u - \mathcal{I}_h u|_{s,\tau;\mathcal{T}_H} = |\hat{u} - \mathcal{I}_h \hat{u}|_{s,\tau;\mathcal{T}_H} \leq |\hat{u}|_{s,\tau;\mathcal{T}_H} + |\mathcal{I}_h \hat{u}|_{s,\tau}.$$

By the triangle inequality,

$$|\mathcal{I}_h \hat{u}|_{s,\tau} = |\sum F_i(\hat{u}) \phi_i|_{s,\tau} \leq \sum |F_i(\hat{u})| |\phi_i|_{s,\tau}.$$

The above sums are taken over the nodes of τ with $\alpha_i < m$. From the definition of F_i and Assumption 3.2, we see that

$$|F_i(\hat{u})| \leq C h_{\tau}^{-1} |\hat{u}|_{\alpha_i, \tilde{N}; \mathcal{T}_H}.$$

Applying Assumption 3.3 and (3.16) then gives

$$(3.18) \quad |\mathcal{I}_h \hat{u}|_{s,\tau} \leq C \sum_{j < m} h_{\tau}^{j-s} |\hat{u}|_{j, \tilde{N}; \mathcal{T}_H}.$$

Applying Lemma 3.1 with respect to the space V_H gives

$$(3.19) \quad |u - \mathcal{I}_h u|_{s,\tau;\mathcal{T}_H} \leq C h_{\tau}^{m-s} |\hat{u}|_{m, \tilde{N}; \mathcal{T}_H} = C h_{\tau}^{m-s} |u|_{m, \tilde{N}; \mathcal{T}_H}.$$

It follows from (3.19) and the triangle inequality that

$$(3.20) \quad h_\tau^{-2m} |\mathcal{I}_h u - u|_{0,\tau}^2 + |\mathcal{I}_h u|_{m,\tau}^2 \leq C |u|_{m,\tilde{N};\mathcal{T}_H}^2.$$

This completes the analysis for a neighborhood \tilde{N} which does not intersect $\partial\Omega$.

Let τ now be an triangle for which the associated neighborhood \tilde{N} intersects $\partial\Omega$. From the construction of \tilde{N} , there is an edge Γ of \tilde{N} which lies on $\partial\Omega$. Replacing \hat{u} by u in the above argument shows that (3.20) holds for every $\tau \in \mathcal{T}_h$. The inequality (3.17) follows by summation. This completes the proof of the theorem. \square

The above theorem shows that we can achieve simultaneous approximation to functions in a space V_H by a space V_h provided that the two spaces have comparable local mesh size. The local mesh size restriction can be removed in the case when the coarser space V_H is conforming. That \mathcal{I}_h defined by (3.15) satisfies the stable approximation condition is the result of the next theorem.

THEOREM 3.3. *Assume that V_H is contained in $H_0^m(\Omega)$. In addition, assume that for $\tau_H \in \mathcal{T}_H$ and $\tau_h \in \mathcal{T}_h$ with $\tau_H \cap \tau_h \neq \emptyset$,*

$$(3.21) \quad h_{\tau_h} \leq C_1 h_{\tau_H}.$$

Then there exists a constant C such that for every $u \in V_H$,

$$\sum_{\tau \in \mathcal{T}_h} h_\tau^{-2m} |\mathcal{I}_h u - u|_{0,\tau}^2 + |\mathcal{I}_h u|_{m,\Omega;\mathcal{T}_h}^2 \leq C |u|_{m,\Omega}^2.$$

Proof. The proof follows the proof of Theorem 3.2. First we consider triangles $\tau \in \mathcal{T}_h$ which do not intersect the boundary. For such triangles, \mathcal{I}_h reproduces polynomials in \mathcal{P}_{m-1} . We define $p \in \mathcal{P}_{m-1}$ so that

$$\int_\tau D^\beta (u - p) \, dx = 0 \quad \text{for all } |\beta| < m$$

and set $\hat{u} = u - p$. As in the proof of Theorem 3.2, for $s = 0$ or $s = m$,

$$(3.22) \quad |u - \mathcal{I}_h u|_{s,\tau} \leq |\hat{u}|_{s,\tau} + |\mathcal{I}_h \hat{u}|_{s,\tau} \leq C h_\tau^{m-s} |u|_{m,\tau} + |\mathcal{I}_h \hat{u}|_{s,\tau}.$$

By the support properties of the basis functions,

$$|\mathcal{I}_h \hat{u}|_{s,\tau} = |\sum F_i(\hat{u}) \phi_i|_{s,\tau} \leq \sum |F_i(\hat{u})| |\phi_i|_{s,\tau}.$$

The sums above are over the nodes of τ with $\alpha_i < m$. It follows from the Taylor theorem that

$$|F_i(\hat{u})| \leq |\hat{u}|_{\alpha_i,\infty,\tau} \leq C h_\tau^{m-\alpha_i} |u|_{m,\infty,\tau}.$$

Applying Assumption 3.3 gives

$$(3.23) \quad |u - \mathcal{I}_h u|_{s,\tau} \leq C h_\tau^{m-s} (|u|_{m,\tau} + h_\tau |u|_{m,\infty,\tau}).$$

We next show that (3.23) still holds for triangles τ which intersect the boundary. Let $\bar{\mathcal{I}}_h$ denote the interpolation operator which is defined as in (3.15) but includes boundary nodes of degree less than m . In general this operator takes functions into a space which is larger than V_h . However, by Remark 3.2, $\mathcal{I}_h u$ and $\bar{\mathcal{I}}_h u$ are identical for $u \in V_H$. The above argument shows that

$$|u - \bar{\mathcal{I}}_h u|_{s,\tau} \leq C h_\tau^{m-s} (|u|_{m,\tau} + h_\tau |u|_{m,\infty,\tau})$$

and hence (3.23) holds for all triangles of \mathcal{T}_h . By summation,

$$\begin{aligned} \sum_{\tau \in \mathcal{T}_h} h_\tau^{2s-2m} |u - \mathcal{I}_h u|_{s,\tau}^2 &\leq C (|u|_{m,\Omega}^2 + \sum_{\tau \in \mathcal{T}_h} h_\tau^2 |u|_{m,\infty,\tau}^2) \\ &\leq C (|u|_{m,\Omega}^2 + \sum_{\tau_H \in \mathcal{T}_H} h_{\tau_H}^2 |u|_{m,\infty,\tau_H}^2). \end{aligned}$$

Applying Assumption (3.2) then gives

$$\sum_{\tau \in \mathcal{T}_h} h_\tau^{2s-2m} |u - \mathcal{I}_h u|_{s,\tau}^2 \leq C |u|_{m,\Omega}^2.$$

The theorem follows from the above inequality and the triangle inequality. \square

The above theorem shows that Condition (c) holds for Theorem 2.2 when V_H is conforming and (3.21) holds. For this case, Condition (b) follows from the following approximation result.

THEOREM 3.4. *Assume that the hypotheses of Theorem 3.3 hold. Then there exists a constant C such that for any $u \in V_h$ there exists $\chi \in V_H$ satisfying*

$$\sum_{\tau_H \in \mathcal{T}_H} h_{\tau_H}^{-2m} |u - \chi|_{0,\tau_H}^2 + |\chi|_{m,\Omega}^2 \leq C |u|_{m,\Omega;\mathcal{T}_h}^2.$$

Proof. The result when both V_h and V_H are conforming was given in [29]. In the case when V_h is nonconforming, we introduce an intermediate conforming space \tilde{V}_h which is defined with respect to the mesh \mathcal{T}_h using any conforming finite element satisfying the assumptions of this section. Let $\tilde{\mathcal{I}}_h$ denote the connection operator mapping V_h into \tilde{V}_h defined by (3.15). Let u be in V_h . By Theorem 3.2,

$$\sum_{\tau \in \mathcal{T}_h} h_\tau^{-2m} |u - \tilde{\mathcal{I}}_h u|_{0,\tau}^2 + |\tilde{\mathcal{I}}_h u|_{m,\Omega}^2 \leq C |u|_{m,\Omega;\mathcal{T}_h}^2.$$

Since $\tilde{\mathcal{I}}_h u \in \tilde{V}_h \subset H_0^{2m}(\Omega)$, Lemma 3.1 in [29] shows that there is a $\chi \in V_H$ such that

$$\sum_{\tau_H \in \mathcal{T}_H} h_{\tau_H}^{-2m} |\tilde{\mathcal{I}}_h u - \chi|_{0,\tau_H}^2 + |\chi|_{m,\Omega}^2 \leq C |\tilde{\mathcal{I}}_h u|_{m,\Omega}^2.$$

The theorem follows from the above two inequalities and obvious manipulations. \square

4. Second order applications. In this section, we consider the application of the general results of the previous two sections to the solution of the discrete systems which arise from second order elliptic boundary value problems. Although there has been much theoretical work on two-level preconditioners for problems with quasiuniform meshes, less has been done for nonquasiuniform meshes. We start by reviewing some published results and then indicate how the theory of the earlier sections covers more general applications.

Let Ω be a polygonal domain in \mathbf{R}^2 and consider the second order boundary value problem

$$\begin{cases} -\sum_{i,j=1}^2 \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The matrix $[a_{ij}(x)]$ is assumed to be symmetric, uniformly positive definite and bounded in Ω . The weak formulation is: Find $u \in H_0^1(\Omega)$ such that

$$A(u, \chi) = (f, \chi), \quad \text{for all } \chi \in H_0^1(\Omega).$$

Here the associated form $A(\cdot, \cdot)$ is given by

$$A(u, v) = \sum_{i,j=1}^2 \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx$$

and (\cdot, \cdot) denotes the inner product in $L^2(\Omega)$.

The simplest two-level application involves conforming subspaces. Let \mathcal{T}_H be a triangulation of Ω and \mathcal{T}_h be a somewhat finer refinement of \mathcal{T}_H . Let the conforming finite elements be such that the corresponding subspace V_H is contained in V_h . In this case, $A_h(\cdot, \cdot)$ and $A_H(\cdot, \cdot)$ are given by $A(\cdot, \cdot)$ and $(\cdot, \cdot)_h$ and $(\cdot, \cdot)_H$ are given by the $L^2(\Omega)$ inner product on Ω . In addition \mathcal{I}_h is taken to be the natural imbedding. The two-level results are already contained in [14, 15]. The two-level result is also an instance of the multilevel result given in [8] and is valid even in some applications where the meshes are not quasiuniform.

We will consider the generalizations of the algorithm to the cases when \mathcal{T}_H is not a refinement of \mathcal{T}_h and/or V_H and V_h are possibly defined by different finite elements. To this end, we first discuss the smoother M_h .

4.1. Smoother. To describe the smoother, we assume that domain Ω is covered by a collection of overlapping subdomains $\{\Omega_k\}_{k=1}^N$ with $\partial\Omega_k$ aligned with the boundaries of certain $\tau \in \mathcal{T}_h$. In addition, we assume that $\{\Omega_k\}$ satisfies the following conditions:

(C.1) *Finite covering property*: Any point of Ω is covered by $\{\Omega_k\}$ at most N_c times.

(C.2) *Uniform H -overlap*: There exist a partition of unity $\{\theta_k\}$ with $\theta_k \in C_0^\infty(\Omega_k)$ such that

$$\sum_k \theta_k = 1, \quad |\theta_k|_{s,\infty} \leq C/H_k^s, \quad s = 0, 1.$$

The constant N_c is the maximum number of Ω_j which intersect with Ω_i and is assumed to be uniformly bounded independent of the number of subdomains N . The parameter H_k is the diameter of Ω_k . Examples of constructions satisfying conditions (C.1) and (C.2) can be found in [14, 15].

Let V_h^k be the space of functions in V_h whose supports are in $\overline{\Omega}_k$. Let Q_h^k be the (\cdot, \cdot) projector onto V_h^k and P_h^k the $A_h(\cdot, \cdot)$ projector onto V_h^k . Define $A_h^k : V_h^k \mapsto V_h^k$ by

$$(A_h^k v, w) = A_h(v, w) \quad \text{for all } v, w \in V_h^k.$$

The additive Schwarz smoother is defined by

$$(4.1) \quad M_{\text{as}}^{-1} = \sum_k (A_h^k)^{-1} Q_h^k = \sum_k P_h^k A_h^{-1},$$

and the symmetric multiplicative Schwarz smoother is given by

$$(4.2) \quad M_{\text{sms}}^{-1} = \{I - (I - P_h^1) \cdots (I - P_h^{N-1}) \cdot (I - P_h^N)(I - P_h^{N-1}) \cdots (I - P_h^1)\} A_h^{-1}.$$

The block Jacobi and symmetric block Gauss-Seidel methods are special cases of (4.1) and (4.2) respectively.

For this example, the weighted norm is defined by

$$(4.3) \quad \|v\|_\Lambda = \left(\sum_k H_k^{-2} \|v\|_{\Omega_k}^2 \right)^{1/2}.$$

LEMMA 4.1. *Assume that Conditions (C.1) and (C.2) and Assumptions 3.1–3.4 hold. Then the smoothers M_{as} and M_{sms} satisfy*

$$\begin{aligned} N_c^{-1} A_h(u, u) &\leq (M_{\text{as}} u, u) \leq C[\|u\|_\Lambda^2 + A_h(u, u)], & \text{for all } u \in V_h, \\ A_h(u, u) &\leq (M_{\text{sms}} u, u) \leq C[\|u\|_\Lambda^2 + A_h(u, u)], & \text{for all } u \in V_h. \end{aligned}$$

The constant C above is independent of the mesh parameter.

Proof. The lower bounds are straightforward. We establish the upper bounds for M_{as} first. It is well known that an additive operator of the form of (4.1) is characterized by the identity

$$(M_{\text{as}}u, u)_h = \inf_{u_k \in V_k, \sum u_k = u} \sum A_h(u_k, u_k).$$

Consequently, it suffices to construct a decomposition $u = \sum u_k$ with $u_k \in V_h^k$ such that

$$(4.4) \quad \sum A_h(u_k, u_k) \leq C(\|u\|_\Lambda^2 + A_h(u, u)).$$

Denote by Π_h the nodal interpolation operator onto V_h . Let $\{\theta_k\}$ be the partition of unity satisfying (C.2) and let $u_k = \Pi_h v_k$ where $v_k = \theta_k u$. Note that u_k is in V_h^k and, since Π_h is linear, $u = \sum u_k$.

Let τ be a triangle in \mathcal{T}_h which does not intersect $\partial\Omega$. By the properties of Π_h and the basis functions of V_h , we have that for c equal to the mean value of v_k on τ ,

$$\begin{aligned} |\Pi_h v_k|_{1,\tau} &= |\Pi_h(v_k - c)|_{1,\tau} \leq \sum_{x_i \in \tau} |\mathcal{L}_i(v_k - c)(x_i)| |\phi_i|_{1,\tau} \\ &\leq C \sum_{x_i \in \tau} |v_k - c|_{\alpha_i, \infty, \tau} h_\tau^{\alpha_i}. \end{aligned}$$

Clearly

$$|v_k - c|_{\alpha_i, \infty, \tau} \leq Ch_\tau^{1-\alpha_i} |v_k|_{1, \infty, \tau}$$

holds for $\alpha_i = 0, 1$. Combining the above two inequalities gives

$$|\Pi_h v_k|_{1,\tau} \leq Ch_\tau |v_k|_{1, \infty, \tau}.$$

Using Assumption 3.2 gives

$$(4.5) \quad \begin{aligned} |u_k|_{1,\tau} &\leq h_\tau |\theta_k u|_{1, \infty, \tau} \leq h_\tau (|\theta_k|_{0, \infty} |u|_{1, \infty, \tau} + |\theta_k|_{1, \infty} |u|_{0, \infty, \tau}) \\ &\leq C(|u|_{1,\tau} + \frac{1}{H_k} |u|_{0,\tau}). \end{aligned}$$

For triangles which intersect the boundary we consider the nodal interpolation operator $\tilde{\Pi}_h$ which maps onto the extended finite element space defined by also including the remaining nodes on $\partial\Omega$. It is easy to check that $\tilde{\Pi}_h v_k = \Pi_h v_k$. Applying the above argument to $\tilde{\Pi}_h v_k$ implies that (4.5) holds for all triangles in \mathcal{T}_h .

Squaring (4.5) and summing over Ω_k gives

$$|u_k|_{1, \Omega; \mathcal{T}_h}^2 = |\Pi_h(\theta_k u)|_{1, \Omega_k; \mathcal{T}_h}^2 \leq C \left(|u|_{1, \Omega_k; \mathcal{T}_h}^2 + \frac{1}{H_k^2} |u|_{0, \Omega_k; \mathcal{T}_h}^2 \right)$$

The inequality (4.4) follows by summing over k and using the finite covering assumption (C.1). This proves the theorem for the additive smoother.

The upper estimate for M_{sms} follows from the estimate for M_{as} and the techniques of [6]. Indeed, to prove the upper estimate for the product, it suffices to show that

$$A_h(M_{\text{as}}^{-1}A_h v, v) \leq C_1 A_h(M_{\text{sms}}^{-1}A_h v, v) \quad \text{for all } v \in V_h.$$

From (4.1) and (4.2) we see that the above inequality reduces to a standard inequality involving the sum and product of projectors. Results of this kind were established in [6]. \square

The results in the above lemma, in a slightly different form, are known and have been used to derive results for two-level additive Schwarz methods; cf. [14, 15]. An analysis of a two-level additive Schwarz method, with a symmetric block Gauss-Seidel smoother, was given in [12].

REMARK 4.1. *In the case of $m = 1$, the above proof is valid for nonconforming finite elements as well. It is also valid for $m = 2$ provided that the space is conforming. For $m = 2$ and nonconforming finite elements, the proof has to be modified for the boundary triangles. The weighted norm is given by $\|w\|_{\Lambda}^2 = \sum_k H_k^{-4} \|v\|_{\Omega_k}^2$ for $m = 2$.*

4.2. Preconditioning with comparable meshes. The generalization which we shall consider now relaxes the relationship between the triangulations \mathcal{T}_h and \mathcal{T}_H . The forms $(\cdot, \cdot)_h$, $(\cdot, \cdot)_H$, $A_h(\cdot, \cdot)$ and $A_H(\cdot, \cdot)$ are as above and the operator \mathcal{I}_h is defined by nodal interpolation. If we assume that the two meshes are quasiuniform and that h and H are of comparable size, then regularity and approximation results are known (see, [7, 26]). Thus, the two-level result follows from Theorem 2.1. A regularity free result in this case was also given in [27]. Applying the theory of the previous sections, we will extend these results to the case when the meshes are not assumed to be quasiuniform.

We first consider the case when \mathcal{T}_h and \mathcal{T}_H are comparable (see (3.16)). For smoothers, we can use either block Jacobi or symmetric block Gauss-Seidel smoothers. Partition the nodes into subsets $\{\Omega_h^k\}$ and let $\{V_h^k\}$ denote the linear span of the corresponding nodal basis functions. Let Q_h^k, A_h^k and P_h^k be as in the previous subsection. With the spaces and operators so defined, the block Jacobi and symmetric block Gauss-Seidel smoothers are given by (4.1) and (4.2). As a consequence of Lemma 4.1, the above smoothers satisfy Condition (a).

Let the grid transfer operator $\mathcal{I}_h : V_H \mapsto V_h$ be defined by (3.15). Note that with

this special choice of $\{\Omega_h^k\}$ and note that \mathcal{T}_H and \mathcal{T}_h are comparable,

$$\|w\|_\Lambda^2 \approx \sum_{\tau \in \mathcal{T}_h} h_\tau^{-2} \|w\|_\tau^2 \approx \sum_{\tau \in \mathcal{T}_H} h_\tau^{-2} \|w\|_\tau^2 \quad \text{for all } w \in L^2(\Omega).$$

Theorem 3.2 shows that Condition (c) holds.

Let \mathcal{I}_H denote the grid transfer operator onto V_H . Applying Theorem 3.2 with the roles of V_H and V_h switched, we obtain

$$\|u - \mathcal{I}_H u\|_\Lambda^2 + A_H(\mathcal{I}_H u, \mathcal{I}_H u) \leq C A_h(u, u) \quad \text{for all } u \in V_h.$$

This implies Condition (b).

The following theorem is now a consequence of Theorem 2.2.

THEOREM 4.2. *Let \mathcal{T}_h and \mathcal{T}_H be comparable meshes and V_h and V_H be the corresponding conforming finite element approximation spaces. Let $\mathcal{I}_h : V_H \mapsto V_h$ be the grid transfer operator defined by (3.15) and M_h be given by M_{as} or M_{sms} . Then the two-level operator B_h defined by (2.5) provides a uniform preconditioner for the operator A_h .*

REMARK 4.2. *The elements defining V_h and V_H can be of different type. In addition, the meshes \mathcal{T}_h and \mathcal{T}_H need not align and can be nonquasiuniform as long as they satisfy the local comparability condition; see (3.16).*

REMARK 4.3. *The results of this subsection carry over easily to nonconforming finite element approximation. This includes applications where one or both of the spaces V_h and V_H are nonconforming. In these cases, one replaces the form $A(\cdot, \cdot)$ by the discrete form, e.g., if V_h is nonconforming, then*

$$A_h(u, v) = \sum_{\tau \in \mathcal{T}_h} \sum_{i,j=1}^2 \int_\tau a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx.$$

4.3. Preconditioning with a significantly coarser mesh. We now discuss the case when \mathcal{T}_H is genuinely coarser than \mathcal{T}_h . By this we mean that there are triangles τ_h and τ_H with $\tau_h \cap \tau_H \neq \emptyset$ and τ_h significantly smaller than τ_H so that (3.16) holds only for extremely small C_0 . In this subsection, we will require that V_H be a conforming approximation.

Unlike the previous example, to get an optimal convergence rate, we cannot use simple smoothers like block Jacobi or Gauss-Seidel. We will use smoothers based on the one-level overlapping Schwarz methods defined by (4.1) and (4.2). Thus, the overall iterative procedure can be thought of as a two-level Schwarz method on

(possibly) nonnested meshes. As we shall see, smoothers with local overlapping of size h_{τ_H} suffice.

We assume that \mathcal{T}_H is comparable with $\{\Omega_k, H_k\}$ in the sense that if $\tau_H \in \mathcal{T}_H$ and $\tau_H \cap \Omega_k \neq \emptyset$ then

$$(4.6) \quad C_0 h_{\tau_H} \leq H_k \leq C_1 h_{\tau_H}.$$

Let V_H be a conforming finite element space defined with respect to \mathcal{T}_H and \mathcal{I}_h be defined by (3.15). The weighted norm $\|\cdot\|_\Lambda$ is given by (4.3).

Lemma 4.1 shows that smoothers defined by (4.1) and (4.2) satisfy Condition (a). It follows from Theorem 3.3 that \mathcal{I}_h satisfies Condition (c). Condition (b) follows from Theorem 3.4. The following theorem is now a consequence of Theorem 2.2.

THEOREM 4.3. *Let the spaces V_h and V_H be as above and \mathcal{I}_h be the grid transfer operator defined by (3.15). In addition, let the smoothing operator be given by (4.1) or (4.2) with subspaces defined in terms of the overlapping cover $\{\Omega_k\}$ satisfying Conditions (C.1) and (C.2). Then the two-level operator B_h defined by (2.5) provides a uniform preconditioner for the operator A_h .*

REMARK 4.4. *As in the previous theorem, the two spaces can be defined from different types of finite elements and the meshes need not align. The space V_h may be nonconforming.*

5. Application to the biharmonic Dirichlet problem. In this section, we apply the theory of Sections 2 and 3 to the biharmonic Dirichlet problem. As these theorems and their proofs are similar to those of the previous section, we give them without proof. Both [23] and [27] give some results for two grid preconditioners applied to the biharmonic problem. The results which we shall give in this section are stronger in that they apply to approximations with meshes which are not quasiuniform.

Let Ω be a polygonal domain in \mathbf{R}^2 and consider the biharmonic Dirichlet problem with homogeneous boundary conditions:

$$\begin{cases} \Delta^2 u &= f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} &= 0 & \text{on } \partial\Omega. \end{cases}$$

A weak formulation is to find $u \in H_0^2(\Omega)$ such that

$$A(u, v) = (f, v), \quad \forall v \in H_0^2(\Omega)$$

where (\cdot, \cdot) is the L_2 inner product,

$$(5.1) \quad A(u, v) = \sigma(\Delta u, \Delta v)_{L^2} + (1 - \sigma)(u, v)_{H^2},$$

and $0 < \sigma < \frac{1}{2}$ is *Poisson's ratio*. Here

$$(u, v)_{H^2} \equiv \int_{\Omega} \left(\frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_1^2} + 2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_2^2} \right) dx.$$

The canonical example of a conforming finite element space is the Argyris element [1]. This space consists of piecewise polynomials of degree five and thus has twenty one degrees of freedom. At each node of the triangle, there is a nodal function corresponding to point value, $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$, $\frac{\partial^2}{\partial x_1^2}$, $\frac{\partial^2}{\partial x_2^2}$, and $\frac{\partial^2}{\partial x_1 \partial x_2}$. In addition, the center of the edges are nodes associated with the normal derivative at those points.

Other conforming examples on triangles include the Bell triangle, Hsieh-Clough-Tocher, reduced Hsieh-Clough-Tocher triangles, the singular Zienkiewicz triangle, and the reduced singular Zienkiewicz triangle. Our theorems apply to all of these examples. Note that the singular Zienkiewicz elements contain functions which are not polynomials. We take $A_h(v, w)$ and $A_H(\cdot, \cdot)$ to be $A(\cdot, \cdot)$.

The simplest nonconforming finite element is the Morley element. For this example, functions are piecewise quadratic with nodal values consisting of the functional values at the vertices and the normal derivative values at the center of the edges. The Fraeijs de Veubeke triangle provides another example of a nonconforming finite element for the biharmonic problem.

Let \mathcal{T}_h and \mathcal{T}_H be triangulations of Ω and V_h and V_H be the corresponding finite element spaces associated with possibly different elements of the type mentioned above. The meshes \mathcal{T}_h and \mathcal{T}_H need not align and can be nonquasiuniform as long as they satisfy the local comparability condition (see, (3.16)). In the case of nonconforming spaces, e.g., if V_h is nonconforming, we take

$$(5.2) \quad A_h(u, v) = \sum_{\tau \in \mathcal{T}_h} [\sigma(\Delta u, \Delta v)_{L^2(\tau)} + (1 - \sigma)(u, v)_{H^2(\tau)}].$$

Here $L^2(\tau)$ and $H^2(\tau)$ indicate the forms defined by integration only over τ .

As usual, we define the operator \mathcal{I}_h by (3.15) and set

$$\|w\|_{\Lambda} = \left(\sum_{\tau \in \mathcal{T}_h} h_{\tau}^{-4} \|w\|_{\tau}^2 \right)^{1/2} \quad \text{for all } w \in L^2(\Omega).$$

For smoothing operators, we take the operators corresponding to block Jacobi or Gauss-Seidel iterations schemes. That these smoothers satisfy Condition (a) follows as in Lemma 4.1. We have the following theorem.

THEOREM 5.1. *Let \mathcal{T}_h and \mathcal{T}_H be comparable meshes and V_h and V_H be biharmonic finite element approximation spaces as discussed above. Let \mathcal{I}_h be the nodal*

interpolation operator defined by (3.15) and M_h be defined to be the block Jacobi or the symmetric block Gauss-Seidel operator (M_{as} or M_{sms}). Then the two-level operator B_h defined by (2.5) provides a uniform preconditioner for the operator A_h defined from the form given by (5.2).

The case when one uses a significantly coarser triangulation \mathcal{T}_H is similar to that considered in Subsection 4.3. We must again use smoothing based on an overlapping Schwarz method. In this case the partition functions θ_k are constructed to be in $C_0^\infty(\Omega_k)$ and satisfy

$$|\theta_k|_{s,\infty} \leq CH_k^{-s}, \text{ for } s = 0, 1, 2,$$

where H_k is the overlap parameter. The norm $\|\cdot\|_\Lambda$ is defined by

$$\|v\|_\Lambda = \left(\sum_k H_k^{-4} \|v\|_{\Omega_k}^2 \right)^{1/2}.$$

The coarse triangulation \mathcal{T}_H is assumed to be comparable with $\{\Omega_k, H_k\}$ in the sense of (4.6) and the coarse space is assumed to be conforming. Applying Lemma 4.1 and Theorems 3.3 and 3.4 shows that Theorem 5.1 holds for this application as well; cf. [29] for the case in which both V_h and V_H are conforming.

In the case when \mathcal{T}_H is significantly coarser than \mathcal{T}_h and the coarse space V_H is nonconforming, the above theorem does not provide a result for the algorithm if the the grid transfer operator \mathcal{I}_h is defined by (3.15). To get an optimal result for the algorithm in such a case it is necessary to use a grid transfer operator that is more sophisticated than the one defined by (3.15); cf. [11].

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