UNIFORM CONVERGENCE OF THE MULTIGRID V-CYCLE FOR AN ANISOTROPIC PROBLEM

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Abstract

In this paper, we consider the linear systems arising from the standard finite element discretizations of certain second order anisotropic problems. We study the performance of a V-cycle multigrid method applied to the finite element equations. Since the usual "regularity and approximation" assumption does not hold for the anisotropic finite element problems, the standard multigrid convergence theory cannot be applied directly. In this paper, a modification of the theory of Braess and Hackbusch will be presented. We show that the V-cycle multigrid iteration with a line smoother is a uniform contraction in the energy norm. In the verification of the hypotheses in our theory, we use a weighted L^2-norm estimate for the error in the Galerkin finite element approximation and a smoothing property of the line smoothers which is proved in this paper.

1 Introduction

The purpose of this paper is to study the V-cycle multigrid methods for certain second order anisotropic finite element problems. The convergence properties of the V-cycle multigrid method for second order selfadjoint elliptic finite element equations are well understood in the cases in which the differential operators are uniformly bounded and elliptic; cf. Braess and Hackbusch [1], Bramble and Pasciak [3, 4], Bramble, Pasciak, Wang and Xu [5] and the references in these papers. The common ingredient in the analysis is the so-called "regularity and approximation" condition. The success of the multigrid methods in these cases is due to the fact that the smoothers are effective in reducing the nonsmooth components of the error and the coarse grid corrections are effective in reducing the smooth components. In this paper, we shall establish a convergence theory for the standard V-cycle multigrid algorithm for anisotropic equations defined on the unit square. We shall consider finite element approximations to this problem. For the anisotropic problem considered in this paper, the standard finite element solution has a "poor" approximation property and hence the coarse grid solves in the multigrid algorithm are not effective in reducing the smooth components of the errors. This is in contrast to the cases in which the differential operators are uniformly bounded and elliptic. When a Jacobi or a Gauss-Seidel smoother is used, the multigrid algorithm does not provide a uniform reduction in the error. The remedy is to use a smoother, such as a line Jacobi or a line Gauss-Seidel smoother, that is effective in reducing components of the error in a larger part of the spectrum.

Since the usual "regularity and approximation" condition does not hold in this case, the V-cycle multigrid theory of Braess and Hackbusch [1] and Bramble and

Pasciak [3] cannot be applied directly. A modification of the theory of Braess and Hackbusch will be presented. In the verification of the hypotheses of the theory, we use a weighted L^2-norm error estimate for the finite element approximation and a smoothing property of the line Jacobi and the line Gauss-Seidel smoothers.

Model problem. Let $\Omega = (0,1)^2$ be the unit square and consider the equation

$$\begin{cases} Au = -[(au_x)_x + (bu_y)_y] = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where, $a(x,y)$ and $b(x,y)$ are positive functions.

We are interested in the cases in which $a(x,y)$ is of unit size and $b(x,y)$ is possibly small. More precisely, we assume that $a(x,y)$ is uniformly bounded from above and below and $b(x,y)$ is bounded uniformly from above with

$$0 < a_{\min} \le a(x,y) \le a_{\max}, \tag{1.2}$$

and

$$0 < \qquad b(x,y) \le b_{\max}. \tag{1.3}$$

We do not assume, however, that $b(x,y)$ has a uniform positive lower bound.

To carry out our analysis for the multigrid algorithm, however, we shall also make the following technical assumptions on the coefficients. We assume that certain first derivatives of $a(x,y)$ and $b(x,y)$ are uniformly bounded in the following sense:

$$\frac{|\nabla a|}{a} \le \frac{|\nabla a|}{a_{\min}} \le C_a, \quad \text{and} \quad \frac{|b_y|}{b} \le C_b. \tag{1.4}$$

Since $a(x,y)$ is assumed to be uniformly bounded above and below in (1.2), the first inequality in (1.4) is the same as

$$|\nabla a| \le C.$$

On the other hand, since $b(x,y)$ is not assumed to have a uniform positive lower bound, the second inequality in (1.4) states that that $b(x,y)$ does not change very much in the y direction relative to its magnitude. This condition can also be written as

$$\begin{cases} b(x,y) = \epsilon(x)\tilde{b}(x,y) \quad \text{with} \\ \tilde{c}_1 \le \tilde{b}(x,y) \le \tilde{c}_2 \quad \text{and} \quad |\tilde{b}_y(x,y)| \le \tilde{c}_3. \end{cases} \tag{1.5}$$

Clearly, (1.5) implies the second inequality in (1.4). Conversely, if we set $\epsilon(x) = \max_y b(x,y)$ and $\tilde{b}(x,y) = b(x,y)/\epsilon(x)$, then the second inequality in (1.4) implies that (1.5) holds with $e^{-C_b} \le \tilde{b}(x,y) \le 1$ and $|\tilde{b}_y(x,y)| \le C_b$. In our subsequent analysis, the estimate for the rate of convergence of the V-cycle multigrid algorithm will depend on the constants in (1.2), (1.3) and (1.4), but not on a positive lower bound for $b(x,y)$. We will often use (1.5) instead of the second inequality in (1.4). Without loss of generality, we can assume $\epsilon(x) \leq 1$ in (1.5).

The weak form of (1.1) is the following: Find $u \in H_0^1(\Omega)$ such that

$$A(u,\phi) = (f,\phi), \quad \text{for all} \quad \phi \in H_0^1(\Omega), \tag{1.6}$$

where

$$A(u,v) = \int_\Omega \left[a(x,y)u_x v_x + b(x,y)u_y v_y\right] \mathrm{d}x\mathrm{d}y.$$

Here $(\cdot,\cdot)$ is the L^2 inner product. We set $\|\cdot\|_A = A(\cdot,\cdot)^{1/2}$, the "energy norm". We shall prove a uniform convergence estimate for the V-cycle multigrid algorithm for solving the finite element equations approximating (1.6).

The remainder of the paper is organized as follows. In §2 we introduce the standard V-cycle multigrid algorithm and provide a modification of the convergence theory of Braess and Hackbusch [1]. In §3, we prove an *a priori* estimate for the solutions of the anisotropic problem. Standard finite element approximations to the anisotropic problem are considered in §4. An approximation property of the Galerkin projection is proved. A weighted L^2-norm error estimate is then established by using the regularity result proved in §3 and the duality argument of Aubin and Nitsche. The smoothing properties of the line Jacobi and the line Gauss-Seidel smoothers are formulated and proved in §5. In §6 we apply the theory of §2 to the anisotropic finite element problem. It is shown that the V-cycle multigrid method with a line smoother is a uniform contraction in the energy norm $\|\cdot\|_A$. This convergence result is based on the approximation property of Galerkin projection and the smoothing property of the line Jacobi and the line Gauss-Seidel methods. Finally, in §7, we formulate the multigrid algorithm in terms of vectors and matrices.

2 Multigrid algorithm and theory

In this section, we consider the standard V-cycle multigrid algorithm and provide a modification of the multigrid convergence theory of Braess and Hackbusch [1]. To this end, we consider a sequence of nested finite element spaces

$$M_1 \subset M_2 \subset \cdots \subset M_J.$$

The finite element problem on M_k is the following: Find $u_k \in M_k$ such that

$$A(u_k,\phi) = (f,\phi), \quad \text{for all} \quad \phi \in M_k.$$

The L^2 projection $Q_k : L^2 \to M_k$ and the Galerkin projection $P_k : H_0^1 \to M_k$ are defined by

$$(Q_k w,\phi) = (w,\phi), \quad \text{for all} \quad \phi \in M_k,$$

and

$$A(P_k w,\phi) = A(w,\phi), \quad \text{for all} \quad \phi \in M_k.$$

Let $A_k : M_k \to M_k$ be defined by

$$(A_k w, \phi) = A(w, \phi), \quad \text{for all} \quad \phi \in M_k.$$

Then the finite element equations can be rewritten in the form

$$A_k u_k = f_k := Q_k f.$$

To define the multigrid algorithm, we need smoothing operators $R_k : M_k \to M_k$. We shall denote by R_k^t the adjoint of R_k with respect to the inner product $(\cdot, \cdot)$. Properties required of the smoothers will be stated later when needed.

Given an initial iterate $u^0 \in M_k$, a linear multigrid algorithm produces a sequence of approximations to $u_k = A_k^{-1} f_k$ as

$$u^{m+1} = \mathrm{Mg}_k(u^m, f_k) \equiv u^m + B_k(f_k - A_k u^m), \quad m = 0, 1, \ldots . \tag{2.1}$$

The multigrid process $\mathrm{Mg}_k(\cdot, \cdot)$ (or equivalently B_k) is defined recursively as follows.

Algorithm 2.1 *With u^0 and $g \in M_1$, set $\mathrm{Mg}_1(u^0, g) = A_1^{-1} g$ (or equivalently $B_1 = A_1^{-1}$). For $k > 1$ and u^0 and $g \in M_k$, $u^1 = \mathrm{Mg}_k(u^0, g)$ is defined as follows:*

(1) *Pre-smoothing*: $u^{1/3} = u^0 + R_k^t(g - A_k u^0)$.

(2) *Coarse grid correction*:

$$\begin{aligned} u^{2/3} &= u^{1/3} + \mathrm{Mg}_{k-1}(0, Q_{k-1}(g - A_k u^{1/3})) \\ &= u^{1/3} + B_{k-1} Q_{k-1}(g - A_k u^{1/3}) \end{aligned}$$

(3) *Post-smoothing*: $u^1 = u^{2/3} + R_k(g - A_k u^{2/3})$.

To understand the multigrid algorithm, we first discuss briefly the smoothing operator, R_k. Given a smoother, R_k, the solution of $A_k u = f_k$ can be computed iteratively by the linear iteration

$$x^{m+1} = x^m + R_k(f_k - A_k x^m), \quad m = 0, 1, 2, \ldots . \tag{2.2}$$

The error propagation operator is $K_k = I - R_k A_k$ and the error, $e^m \equiv A_k^{-1} f_k - x^m$, satisfies

$$e^{m+1} = K_k e^m.$$

We assume that the above linear iteration is a contraction in the norm $\| \cdot \|_A$, i.e.,

$$\|K_k\|_A = \sup_{\|v\|_A = \|w\|_A = 1} A(K_k v, w) < 1.$$

Set $K_k^* = (I - R_k^t A_k)$. Then K_k^* is the adjoint of K_k with respect to the inner product $A(\cdot, \cdot)$ and $K_k^* K_k$ is self-adjoint with respect to the inner product $A(\cdot, \cdot)$. Consequently

$$\|K_k^* K_k\|_A = \|K_k^*\|_A^2 = \|K_k\|_A^2 < 1.$$

A simple manipulation shows that

$$A(K_k v, K_k v) = A(v, v) - (\bar{R}_k A_k v, A_k v),$$

with

$$\bar{R}_k = R_k + R_k^t - R_k^t A_k R_k.$$

Thus, the above assumption on the smoother is equivalent to assuming that $\bar{R}_k$ is positive definite. Note that $\bar{R}_k A_k = I - K_k^* K_k$.

To estimate the rate of convergence of iteration (2.1), with $\mathrm{Mg}_k(\cdot,\cdot)$ defined by Algorithm 2.1, we first derive, as in Bramble and Pasciak [2], a two-level recurrence relation for the error operator of the V-cycle multigrid algorithm. Let $u = A_k^{-1} g$. Define $e^0 = u - u^0$, $e^{1/3} = u - u^{1/3}, e^{2/3} = u - u^{2/3}$ and $e^1 = u - u^1$. Then by the definition of the multigrid algorithm and the above discussion concerning the linear iteration (2.2), we have

$$\begin{aligned} e^{1/3} &= (I - R_k^t A_k) e^0, \\ e^{2/3} &= (I - B_{k-1} Q_{k-1} A_k) e^{1/3} = (I - B_{k-1} A_{k-1} P_{k-1}) e^{1/3}, \\ e^1 &= (I - R_k A_k) e^{2/3} = (I - R_k A_k)(I - B_{k-1} A_{k-1} P_{k-1})(I - R_k^t A_k) e^0. \end{aligned}$$

In the second equation, we have used the identity $Q_{k-1} A_k = A_{k-1} P_{k-1}$. Combining these three equations, we obtain the following recurrence relation:

$$A((I - B_k A_k) v, v) = A((I - B_{k-1} A_{k-1} P_{k-1}) K_k^* v, K_k^* v), \text{ for all } v \in M_k. \quad (2.3)$$

Here $K_k = (I - R_k A_k)$ and $K_k^* = (I - R_k^t A_k)$ are the error propagation operators corresponding to the smoothers R_k and R_k^t, respectively.

Denote by λ_k the largest eigenvalue of A_k. In the standard multigrid convergence theory, the smoother, R_k, is assumed to satisfy the smoothing property

$$\frac{\omega}{\lambda_k}(v, v) \le (\bar{R}_k v, v), \quad \text{for all} \quad v \in M_k.$$

In addition, the following type of "regularity and approximation" condition is used: there exist $\alpha \in (0, 1]$ and $C > 0$, independent of k, such that

$$(A_k^{1-\alpha}(I - P_{k-1}) v, (I - P_{k-1}) v) \le C \lambda_k^{-\alpha} A((I - P_{k-1}) v, v), \quad \text{for all} \quad v \in M_k.$$

This type of regularity and approximation condition, however, does not hold for the anisotropic problem, and therefore, we cannot directly apply the theory of Braess and Hackbusch [1] and Bramble and Pasciak [2, 3]. We shall provide a modification of the theory of Braess and Hackbusch.

We first consider symmetric smoothers. In this case, the condition that $\|K_k\|_A < 1$ is equivalent to the condition that the spectrum $\sigma(K_k) \subset [0, 1)$.

Lemma 2.1 *Assume that R_k is symmetric and that $\|K_k\|_A \equiv \|I - R_k A_k\|_A < 1$. Assume further that there is a constant C_M independent of k such that*

$$(R_k^{-1}(I - P_{k-1}) v, (I - P_{k-1}) v) \le C_M A((I - P_{k-1}) v, v), \ \textit{for all } v \in M_k. \quad (2.4)$$

Then the multigrid algorithm defined in Algorithm 2.1 *satisfies*

$$0 \le A((I - B_k A_k)v, v) \le \delta A(v, v), \quad \textit{for all} \quad v \in M_k,$$

with $\delta = C_M/(2 + C_M)$.

Proof. We will prove by induction that this estimate holds. Clearly the assertion holds for $k = 1$. Suppose now that the assertion holds for $k - 1$, i.e.,

$$0 \le A((I - B_{k-1} A_{k-1})v, v) \le \delta A(v, v). \tag{2.5}$$

Using the recurrence relation (2.3), we have, for $v \in M_k$,

$$\begin{aligned} A((I - B_k A_k)v, v) &= A((I - P_{k-1})K_k^* v, K_k^* v) \\ &\quad + A((I - B_{k-1} A_{k-1})P_{k-1} K_k^* v, P_{k-1} K_k^* v). \end{aligned}$$

It is straightforward to show that the induction hypothesis (2.5) implies that

$$0 \le A((I - B_k A_k)v, v) \le (1 - \delta) A((I - P_{k-1})K_k^* v, K_k^* v) + \delta A(K_k^* v, K_k^* v). \tag{2.6}$$

We now estimate the first term on the right hand side of (2.6). By the Cauchy-Schwarz inequality and the hypothesis (2.4), we have, for $w \in M_k$,

$$\begin{aligned} A((I - P_{k-1})w, w) &\le (R_k^{-1}(I - P_{k-1})w, (I - P_{k-1})w)^{1/2} (R_k A_k w, A_k w)^{1/2} \\ &\le C_M^{1/2} A((I - P_{k-1})w, w)^{1/2} (R_k A_k w, A_k w)^{1/2}. \end{aligned}$$

Cancelling the common factor, we get

$$A((I - P_{k-1})w, w) \le C_M (R_k A_k w, A_k w), \quad \text{for all} \quad w \in M_k.$$

Since R_k is symmetric, $K_k^* = K_k$. Applying the above inequality with $w = K_k v$ and recalling that $R_k A_k = I - K_k$ and that $\sigma(K_k) \in [0, 1)$, we obtain

$$\begin{aligned} A((I - P_{k-1})K_k v, K_k v) &\le C_M (R_k A_k K_k v, A_k K_k v) \\ &= C_M ((I - K_k) K_k v, A_k K_k v) \\ &\le \frac{C_M}{2} [A(v, v) - A(K_k v, K_k v)]. \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we obtain

$$\begin{aligned} A((I - B_k A_k)v, v) &\le (1 - \delta) A((I - P_{k-1})K_k v, K_k v) + \delta A(K_k v, K_k v) \\ &\le \frac{C_M}{2} (1 - \delta)[A(v, v) - A(K_k v, K_k v)] + \delta A(K_k v, K_k v)] \\ &\le \delta A(v, v), \end{aligned}$$

with $\delta = C_M/(2 + C_M)$. □

Lemma 2.1 can be modified to allow the use of nonsymmetric smoothing operators such as the line Gauss-Seidel smoother. Recall that $\bar{R}_k = R_k + R_k^t - R_k^t A_k R_k$.

Lemma 2.2 *Assume that* $\|K_k\|_A \equiv \|I - R_k A_k\|_A < 1$ *and that there is a constant* C_M *independent of* k *such that*

$$(\bar{R}_k^{-1}(I - P_{k-1})v, (I - P_{k-1})v) \leq C_M A((I - P_{k-1})v, v), \quad \textit{for all} \ \ v \in M_k. \quad (2.8)$$

Then the multigrid algorithm defined in Algorithm 2.1 satisfies

$$0 \leq A((I - B_k A_k)v, v) \leq \delta A(v, v), \quad \textit{for all} \quad v \in M_k,$$

with $\delta = C_M/(1 + C_M)$.

Proof. The proof is similar to that of Lemma 2.1. We only give an outline here. As in the proof of Lemma 2.1, the assumption in (2.8) implies that

$$A((I - P_{k-1})w, w) \leq C_M(\bar{R}_k A_k w, A_k w), \quad \text{for all} \quad w \in M_k.$$

Applying the above inequality with $w = K_k^* v$ and recalling that $\bar{R}_k A_k = I - K_k^* K_k$ and that $\sigma(K_k^* K_k) \in [0, 1)$, we obtain

$$\begin{aligned} A((I - P_{k-1})K_k^* v, K_k^* v) &\leq C_M(\bar{R}_k A_k K_k^* v, A_k K_k^* v) \\ &= C_M((I - K_k^* K_k)K_k^* v, A_k K_k^* v) \\ &\leq C_M[A(v, v) - A(K_k^* v, K_k^* v)]. \end{aligned}$$

The rest of the proof is identical to that of Lemma 2.1 □

Remark 2.1 It is not necessary to solve the coarse grid problem exactly. If the approximate coarse grid solution satisfies $0 \leq A((I - B_1 A_1)\phi, \phi) \leq \delta_0 < 1$, for all $\phi \in M_1$, then Lemma 2.1 holds with $\delta = \max(\delta_0, C_M/(2 + C_M))$ and Lemma 2.2 holds with $\delta = \max(\delta_0, C_M/(1 + C_M))$. □

Lemmas 2.1 and 2.2 are "soft". To apply the lemmas, we need to establish (2.4) or (2.8) with C_M independent of k. For example, (2.4) can be proved by combining the approximation property

$$(b(I - P_{k-1})v, v) \leq C h_k^2 A((I - P_{k-1})v, v)$$

and the following smoothing property of the line Jacobi smoother

$$(R_k^{-1} v, v) \leq C[A(v, v) + h_k^{-2}(bv, v)].$$

Here h_k is the mesh parameter. To establish (2.8) for a nonsymmetric smoother such as the line Gauss-Seidel smoother, we replace R_k in the above inequality by $\bar{R}_k$. These properties will be subsequently proved.

3 Regularity of the problem

In this section, we derive an *a priori* estimate for the solutions of the anisotropic equation (1.1). This result will be used in the next section to derive error estimates of the Galerkin finite element approximation in the energy norm and a weighted L^2-norm.

We first note that if $a(x,y)=1$ and $b(x,y)=\epsilon$ is a constant, then, by integration by parts, we have for $u \in H_0^1(\Omega)\cap H^2(\Omega)$

$$\begin{aligned}\int_\Omega (u_{xx}^2 + 2u_{xy}^2 + \epsilon u_{yy}^2)\,\mathrm{d}x\mathrm{d}y &= \int_\Omega (u_{xx}^2 + 2u_{xx}u_{yy} + \epsilon u_{yy}^2)\,\mathrm{d}x\mathrm{d}y \\ &\le \int_\Omega \frac{1}{\epsilon}|Au|^2\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

The following lemma is a generalization of this fact to some variable coefficient cases.

Lemma 3.1 (Regularity) *Let the coefficients $a(x,y)$ and $b(x,y)$ satisfy (1.2)–(1.4). Then the following a priori estimate holds:*

$$\int_\Omega (u_{xx}^2 + 2u_{xy}^2 + bu_{yy}^2)\,\mathrm{d}x\mathrm{d}y \le C\int_\Omega \frac{1}{b}|Au|^2\,\mathrm{d}x\mathrm{d}y.$$

Proof. Integrating by parts gives

$$\begin{aligned}2\int_\Omega \frac{1}{b}(au_x)_x(bu_y)_y\,\mathrm{d}x\mathrm{d}y &= 2\int_\Omega \left((au_x)_x u_{yy} + (au_x)_x\frac{b_y}{b}u_y\right)\mathrm{d}x\mathrm{d}y \\ &= 2\int_\Omega \left((au_x)_y u_{xy} + (au_x)_x\frac{b_y}{b}u_y\right)\mathrm{d}x\mathrm{d}y \\ &= 2\int_\Omega \left(au_{xy}^2 + (a_yu_x)u_{xy} + (au_x)_x\frac{b_y}{b}u_y\right)\mathrm{d}x\mathrm{d}y \\ &\ge \int_\Omega \left\{2au_{xy}^2 - \left(\alpha a u_{xy}^2 + \frac{a_y^2}{\alpha a}u_x^2\right) - \left[\frac{\beta}{b}(au_x)_x^2 + \frac{b}{\beta}\left(\frac{b_y}{b}u_y\right)^2\right]\right\}\mathrm{d}x\mathrm{d}y \\ &= \int_\Omega \left(-\frac{\beta}{b}(au_x)_x^2 + (2-\alpha)au_{xy}^2 - \frac{a_y^2}{\alpha a}u_x^2 - \frac{b_y^2}{\beta b}u_y^2\right)\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where α and β are arbitrary positive constants. As a consequence, we have

$$\begin{aligned}\int_\Omega \frac{1}{b}|Au|^2\,\mathrm{d}x\mathrm{d}y &= \int_\Omega \left(\frac{1}{b}(au_x)_x^2 + \frac{2}{b}(au_x)_x(bu_y)_y + \frac{1}{b}(bu_y)_y^2\right)\mathrm{d}x\mathrm{d}y \\ &\ge \int_\Omega \left(\frac{1-\beta}{b}(au_x)_x^2 + (2-\alpha)au_{xy}^2 + \frac{1}{b}(bu_y)_y^2 - \frac{a_y^2}{\alpha a}u_x^2 - \frac{b_y^2}{\beta b}u_y^2\right)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Let $\gamma = \min(1, 1/b_{\max}) \le 1$. Then

$$\begin{aligned}\int_\Omega \frac{1}{b}(au_x)_x^2\,\mathrm{d}x\mathrm{d}y &\ge \gamma\int_\Omega (au_x)_x^2\,\mathrm{d}x\mathrm{d}y \\ &= \gamma\int_\Omega (au_{xx} + a_xu_x)^2\,\mathrm{d}x\mathrm{d}y \\ &\ge \gamma\int_\Omega \left(\frac{1}{2}a^2u_{xx}^2 - a_x^2u_x^2\right)\mathrm{d}x\mathrm{d}y\end{aligned}$$

and

$$\int_\Omega \frac{1}{b}(bu_y)_y^2 \,\mathrm{d}x\mathrm{d}y = \int_\Omega \frac{1}{b}(bu_{yy} + b_y u_y)^2 \,\mathrm{d}x\mathrm{d}y$$
$$\geq \int_\Omega \left(\frac{b}{2}u_{yy}^2 - \frac{b_y^2}{b}u_y^2\right) \mathrm{d}x\mathrm{d}y.$$

Combining the above estimates, we obtain

$$\begin{aligned}\int_\Omega \frac{1}{b}(Au)^2 \,\mathrm{d}x\mathrm{d}y &\geq (1-\beta)\gamma \int_\Omega \left(\frac{1}{2}a^2 u_{xx}^2 - a_x^2 u_x^2\right) \mathrm{d}x\mathrm{d}y + (2-\alpha)\int_\Omega a u_{xy}^2 \,\mathrm{d}x\mathrm{d}y \\ &\quad + \int_\Omega \left(\frac{b}{2}u_{yy}^2 - \frac{b_y^2}{b}u_y^2\right) \mathrm{d}x\mathrm{d}y - \int_\Omega \left(\frac{a_y^2}{\alpha a}u_x^2 + \frac{b_y^2}{\beta b}u_y^2\right) \mathrm{d}x\mathrm{d}y \\ &= \int_\Omega \left(\frac{1}{2}(1-\beta)\gamma a^2 u_{xx}^2 + (2-\alpha) a u_{xy}^2 + (\frac{b}{2}u_{yy}^2)\right) \mathrm{d}x\mathrm{d}y \\ &\quad - \int_\Omega \left((1-\beta)\gamma a_x^2 + \frac{a_y^2}{\alpha a}\right) u_x^2 \,\mathrm{d}x\mathrm{d}y - \int_\Omega \left(\frac{b_y^2}{b} + \frac{b_y^2}{\beta b}\right) u_y^2 \,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_\Omega (u_{xx}^2 + 2u_{xy}^2 + b u_{yy}^2) \,\mathrm{d}x\mathrm{d}y \\ &\quad \leq C\int_\Omega \left[\frac{1}{2}(1-\beta)\gamma a^2 u_{xx}^2 + (2-\alpha) a u_{xy}^2 + (\frac{b}{2}u_{yy}^2)\right] \mathrm{d}x\mathrm{d}y \\ &\quad \leq C\int_\Omega \frac{1}{b}(Au)^2 \,\mathrm{d}x\mathrm{d}y \\ &\qquad + C\int_\Omega \left[\left((1-\beta)\gamma a_x^2 + \frac{a_y^2}{\alpha a}\right) u_x^2 + \left(\frac{b_y^2}{b} + \frac{b_y^2}{\beta b}\right) u_y^2\right] \mathrm{d}x\mathrm{d}y.\end{aligned}$$

The second integral is clearly bounded by $C\int_\Omega |Au|^2$ and hence bounded by $C\int_\Omega b^{-1}|Au|^2$. This proves the lemma. □

4 Finite element approximation

Let the domain $\Omega = (0,1)^2$ be partitioned into squares with vertices $(ih, jh), h = 1/n$. We consider the linear or the bilinear finite element space M_h associated with this partition. The Galerkin finite element projection $P_h : H_0^1(\Omega) \to M_h$ is defined by

$$A(P_h v, \phi) = A(v, \phi), \quad \text{for all} \quad \phi \in M_h.$$

We need the following results in proving an approximation property of the finite element solutions.

Lemma 4.1 *Let $\mathcal{D} = (0, h_1) \times (0, h_2)$ and E be a side of the rectangular region $\mathcal{D}$. If $v \in H^1(\mathcal{D})$ and $\int_E v \,\mathrm{d}s = 0$, then*

$$\|v\|_{L^2(\mathcal{D})}^2 \leq (h_1^2 \|v_x\|_{L^2(\mathcal{D})}^2 + h_2^2 \|v_y\|_{L^2(\mathcal{D})}^2).$$

Suppose that $b(x,y)=\epsilon(x)\tilde{b}(x,y)$ with $\tilde{c}_1\le\tilde{b}(x,y)\le\tilde{c}_2$, and $\epsilon(x)\le 1$. Denote by E a vertical edge of $\mathcal{D}$. If $v\in H^1(\mathcal{D})$ and $\int_E v\,\mathrm{d}s=0$, then

$$\int_{\mathcal{D}} bv^2\,\mathrm{d}x\mathrm{d}y \le \tilde{c}_2\left(h_1^2\int_{\mathcal{D}} v_x^2\,\mathrm{d}x\mathrm{d}y+\tilde{c}_1^{-1}h_2^2\int_{\mathcal{D}} bv_y^2\,\mathrm{d}x\mathrm{d}y\right).$$

Proof. Without loss of generality, we assume $\int_0^{h_2} u(0,y)\,\mathrm{d}y=0$. Then

$$u(x,y)=u(0,y_0)+\int_0^x u_x(s,y_0)\,\mathrm{d}s+\int_{y_0}^y u_y(x,t)\,\mathrm{d}t.$$

Integrating y_0 from 0 to h_2 and using $\int_0^{h_2} u(0,y_0)\mathrm{d}y_0=0$, we obtain

$$u(x,y)=\frac{1}{h_2}\int_0^{h_2}\mathrm{d}y_0\int_0^x u_x(s,y_0)\,\mathrm{d}s+\frac{1}{h_2}\int_0^{h_2}\mathrm{d}y_0\int_{y_0}^y u_y(x,t)\,\mathrm{d}t.$$

Squaring and then using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|u(x,y)|^2 &\le \frac{2}{h_2^2}\left(\int_0^{h_2}\mathrm{d}y_0\int_0^x |u_x(s,y_0)|\,\mathrm{d}s\right)^2+\frac{2}{h_2^2}\left(\int_0^{h_2}\mathrm{d}y_0\int_0^y |u_y(x,t)|\,\mathrm{d}t\right)^2\\
&\le \frac{2}{h_2^2}h_2x\int_0^{h_2}\mathrm{d}y_0\int_0^{h_1} u_x^2(s,y_0)\,\mathrm{d}s+\frac{2}{h_2^2}h_2^2y\int_0^y u_y^2(x,t)\,\mathrm{d}t\\
&\le \frac{2x}{h_2}\int_{\mathcal{D}} u_x^2\,\mathrm{d}x\mathrm{d}y+2y\int_0^{h_2} u_y^2(x,t)\,\mathrm{d}t. \qquad (4.1)
\end{aligned}$$

Integrating over $\mathcal{D}$, we obtain

$$\int_{\mathcal{D}} |u(x,y)|^2\,\mathrm{d}x\mathrm{d}y\le h_1^2\int_{\mathcal{D}} u_x^2\,\mathrm{d}x\mathrm{d}y+h_2^2\int_{\mathcal{D}} u_y^2\,\mathrm{d}x\mathrm{d}y.$$

This is the first part of the lemma.

We now prove the second part. Using $\epsilon(x)\le 1$, we obtain from (4.1)

$$\epsilon(x)|u(x,y)|^2\le\frac{2x}{h_2}\int_{\mathcal{D}} u_x^2\,\mathrm{d}x\mathrm{d}y+2y\int_0^{h_2}\epsilon(x)u_y^2(x,t)\,\mathrm{d}t.$$

Integrating the above inequality over $\mathcal{D}$ gives

$$\int_{\mathcal{D}}\epsilon(x)v^2\,\mathrm{d}x\mathrm{d}y\le\left(h_1^2\int_{\mathcal{D}} v_x^2\,\mathrm{d}x\mathrm{d}y+h_2^2\int_{\mathcal{D}}\epsilon(x)v_y^2\,\mathrm{d}x\mathrm{d}y\right).$$

Since $\tilde{c}_1\epsilon(x)\le b(x,y)\le\tilde{c}_2\epsilon(x)$, the second part of the lemma follows from the last inequality. □

Using this lemma, we can prove the following error estimate for the nodal value interpolant.

Lemma 4.2 *Let $\pi_h: C(\bar{\Omega})\to M_h$ be the nodal value interpolation operator. Then*

$$\|(I-\pi_h)v\|_A^2\le Ch^2\int_\Omega\left(v_{xx}^2+v_{xy}^2+bv_{yy}^2\right)\mathrm{d}x\mathrm{d}y,\quad \textit{for all}\quad v\in H^2(\Omega).$$

Proof. Let $\tau = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ be arbitrary. Let E_x and E_y be edges in the x and y directions, respectively. For the bilinear element, we have

$$\int_{E_x} (v - \pi_h v)_x \, \mathrm{d}x = \int_{E_y} (v - \pi_h v)_y \, \mathrm{d}y = 0.$$

Applying the first part of Lemma 4.1 to $(v - \pi_h v)_x$ and the second part of Lemma 4.1 to $(v - \pi_h v)_y$ on each element τ, we have

$$\int_\tau a|(v - \pi_h v)_x|^2 \, \mathrm{d}x\mathrm{d}y \le a_{\max} h^2 \int_\tau \left(|(v - \pi_h v)_{xx}|^2 + |(v - \pi_h v)_{xy}|^2 \right) \mathrm{d}x\mathrm{d}y,$$

and

$$\int_\tau b|(v - \pi_h v)_y|^2 \, \mathrm{d}x\mathrm{d}y \le C h^2 \int_\tau \left(|(v - \pi_h v)_{xy}|^2 + b|(v - \pi_h v)_{yy}|^2 \right) \mathrm{d}x\mathrm{d}y.$$

In the bilinear case, $(\pi_h v)_{xx} = (\pi_h v)_{yy} = 0$ in τ and

$$\begin{aligned} \|(\pi_h v)_{xy}\|^2_{L^2(\tau)} &= h^{-2} |[v(x_i, y_j) - v(x_{i-1}, y_j)] - [v(x_i, y_{j-1}) - v(x_{i-1}, y_{j-1})]|^2 \\ &\le \|v_{xy}\|^2_{L^2(\tau)}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_\tau \left(a|(v - \pi_h v)_x|^2 + b|(v - \pi_h v)_y|^2 \right) \mathrm{d}x\mathrm{d}y \\ &\qquad \le C h^2 \int_\tau \left(|(v - \pi_h v)_{xx}|^2 + 2|(v - \pi_h v)_{xy}|^2 + b|(v - \pi_h v)_{yy}|^2 \right) \mathrm{d}x\mathrm{d}y \\ &\qquad \le C h^2 \int_\tau \left(|v_{xx}|^2 + 2|v_{xy}|^2 + b|v_{yy}|^2 \right) \mathrm{d}x\mathrm{d}y. \end{aligned}$$

The result for bilinear elements follows by summing over τ.

In the case of linear elements, we write $\tau = \tau^+ \cup \tau^-$ where τ^+ and τ^- are the two triangles on which $\pi_h v$ is linear. Let ℓ^+ be the linear function on τ which is equal to $\pi_h v$ on τ^+, with ℓ^- similarly defined. Applying the first part of Lemma 4.1 to $(v - \ell^\pm)_x$ and the second part of Lemma 4.1 to $(v - \ell^\pm)_y$ on τ, we obtain

$$\int_\tau a|(v - \ell^\pm)_x|^2 \, \mathrm{d}x\mathrm{d}y \le a_{\max} h^2 \int_\tau \left(|(v - \ell^\pm)_{xx}|^2 + |(v - \ell^\pm)_{xy}|^2 \right) \mathrm{d}x\mathrm{d}y$$

and

$$\int_\tau b|(v - \ell^\pm)_y|^2 \, \mathrm{d}x\mathrm{d}y \le C h^2 \int_\tau \left(|(v - \ell^\pm)_{xy}|^2 + b|(v - \ell^\pm)_{yy}|^2 \right) \mathrm{d}x\mathrm{d}y.$$

Since $\ell^\pm$ is linear, its second derivatives all vanish in τ, and hence we obtain

$$\int_\tau \left(a|(v - \ell^\pm)_x|^2 + b|(v - \ell^\pm)_y|^2 \right) \mathrm{d}x\mathrm{d}y \le C h^2 \int_\tau \left(|v_{xx}|^2 + 2|v_{xy}|^2 + b|v_{yy}|^2 \right) \mathrm{d}x\mathrm{d}y.$$

Consequently,

$$
\begin{aligned}
&\int_\tau \left(a|(v-\pi_h v)_x|^2 + b|(v-\pi_h v)_y|^2\right) \mathrm{d}x\mathrm{d}y \\
&\qquad = \int_{\tau^+} \left(a|(v-\ell^+)_x|^2 + b|(v-\ell^+)_y|^2\right) \mathrm{d}x\mathrm{d}y \\
&\qquad\quad + \int_{\tau^-} \left(a|(v-\ell^-)_x|^2 + b|(v-\ell^-)_y|^2\right) \mathrm{d}x\mathrm{d}y \\
&\qquad \le Ch^2 \int_\tau \left(|v_{xx}|^2 + 2|v_{xy}|^2 + b|v_{yy}|^2\right) \mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

The result now follows by summing over τ. □

The Galerkin projection has the following approximation properties.

Lemma 4.3 *There is a constant C_1 such that for $v \in H_0^1(\Omega) \cap H^2(\Omega)$*

$$
\|(I-P_h)v\|_A^2 \le C_1 h^2 \left(\frac{1}{b}Av, Av\right)
$$

and

$$
(b(I-P_h)v, (I-P_h)v) \le C_1 h^2 \|(I-P_h)v\|_A^2.
$$

Proof. The first inequality is a consequence of Lemma 4.2 and Lemma 3.1, i.e.,

$$
\begin{aligned}
\|v-P_h v\|_A^2 &\le \|v-\pi_h v\|_A^2 \\
&\le Ch^2 \int_\Omega \left(v_{xx}^2 + 2v_{xy}^2 + bv_{yy}^2\right) \mathrm{d}x\mathrm{d}y \\
&\le C_1 h^2 \left(\frac{1}{b}Av, Av\right).
\end{aligned}
$$

For the second we use a duality argument. Let $Aw = b(I-P_h)v$. Then

$$
\begin{aligned}
(b(I-P_h)v, (I-P_h)v) &= A(w, v-P_h v) = A((I-P_h)w, (I-P_h)v) \\
&\le \|(I-P_h)w\|_A \|(I-P_h)v\|_A \\
&\le \sqrt{C_1}h \left(\frac{1}{b}Aw, Aw\right)^{1/2} \|(I-P_h)v\|_A \\
&= \sqrt{C_1}h (b(I-P_h)v, (I-P_h)v)^{1/2} \|(I-P_h)v\|_A.
\end{aligned}
$$

Cancelling the common factor and then squaring, the second inequality follows. □

5 The line Jacobi and Gauss-Seidel smoothers

We consider only linear and bilinear elements. The partition of Ω and the finite element space M_h are defined as in the previous section. To define the line Jacobi and the line Gauss-Seidel smoothers, we introduce a horizontal stripwise decomposition of Ω:

$$
\Omega = \cup \Omega_j, \quad \Omega_j = [(x,y) \in \Omega : (j-1)h < y < (j+1)h].
$$

We partition the finite element space M_h accordingly as

$$M_h = \sum_{j=1}^{n-1} M_{h,j}, \quad \text{where} \quad M_{h,j} = [v \in M_h : v = 0 \text{ in } \Omega \setminus \Omega_j].$$

Note that for linear and bilinear elements this is a direct sum, i.e., the decomposition of $v \in M_h$ as $v = \sum_{j=1}^{n-1} v_j$ with $v_j \in M_{h,j}$ is unique. The operator $A_h : M_h \to M_h$ is defined by

$$(A_h v, \phi) = A(v, \phi), \quad \text{for all} \quad \phi \in M_h.$$

The operator $A_{h,j} : M_{h,j} \to M_{h,j}$, the "restriction" of A_h to $M_{h,j}$, is defined similarly, i.e.,

$$(A_{h,j} v, \phi) = A(v, \phi), \quad \text{for all} \quad \phi \in M_{h,j}.$$

We also need the projection $Q_{h,j} : M_h \to M_{h,j}$ with respect to the L^2 inner product $(\cdot,\cdot)$ and the projection $P_{h,j} : M_h \to M_{h,j}$ with respect to the inner product $A(\cdot,\cdot)$. Note that the relation $P_{h,j} = A_{h,j}^{-1} Q_{h,j} A_h$ holds.

The line Jacobi smoother $\mathfrak{J}_h$ is defined by

$$\mathfrak{J}_h = \sum_{j=1}^{n-1} A_{h,j}^{-1} Q_{h,j}. \tag{5.1}$$

The line Gauss-Seidel smoother $\mathfrak{G}_h$ is defined by

$$\mathfrak{G}_h = [I - (I - P_{h,n-1}) \cdots (I - P_{h,1})] A_h^{-1}, \quad \text{where } P_{h,j} = A_{h,j}^{-1} Q_{h,j} A_h. \tag{5.2}$$

Given $f \in M_h$, $\mathfrak{G}_h f \in M_h$ can be computed as follows:

(i) Set $v_0 = 0$.

(ii) For $i = 1, \ldots, n-1$ define

$$v_i = v_{i-1} + A_{h,i}^{-1} Q_{h,i} (f - A_h v_{i-1}).$$

(iii) Set $\mathfrak{G}_h f = v_{n-1}$.

To establish the smoothing property of the line Jacobi smoother we use the following characterization of $\mathfrak{J}_h$:

$$(\mathfrak{J}_h^{-1} v, v) \equiv \sum_{j=1}^{n-1} A(v_j, v_j), \tag{5.3}$$

where $\sum_j v_j = v$ with $v_j \in M_{h,j}$ (v_j is unique). This result is trivial if we interpret the smoother, $\mathfrak{J}_h$, in (5.1) and (5.3) using a matrix-vector notation. A direct proof (5.3) is also easy. We first note that $(\mathfrak{J}_h v, v) = \sum_j (A_{h,j}^{-1} Q_{h,j} v, Q_{h,j} v) = 0$ implies that $v = 0$. Consequently $\mathfrak{J}_h^{-1}$ exists. Let $v \in M_h$ and $v = \sum_j v_j$ with $v_j \in M_{h,j}$. Since the decomposition is unique, we have $v_j = A_{h,j}^{-1} Q_{h,j} \mathfrak{J}_h^{-1} v$. Equality (5.3) now follows from a simple calculation using the formula for v_j.

A smoothing property of the line Jacobi operator is summarized in the following lemma.

Lemma 5.1 *Let M_h consist of piecewise linear or bilinear functions and let $\mathfrak{J}_h$ be the line Jacobi smoother defined by* (5.1). *Then there is a constant C_2 such that*

$$\frac{1}{2}(Av,v) \le (\mathfrak{J}_h^{-1}v,v) \le C_2\left[(Av,v)+\frac{1}{h^2}(bv,v)\right], \quad \text{for all} \quad v \in M_h.$$

Proof. In either case, we write $v \in M_h$ as $v = \sum_j v_j$, with $v_j \in M_{h,j}$. Recall that this decomposition of v is unique. Let $S_j = [(x,y) \in \Omega : (j-1)h \le y \le jh]$. Then by (5.3)

$$(\mathfrak{J}_h^{-1}v,v) \equiv \sum_{j=1}^{n-1} A(v_j,v_j) = \sum_{j=1}^{n}[A_{S_j}(v_{j-1},v_{j-1}) + A_{S_j}(v_j,v_j)].$$

On strip S_j: $v = v_{j-1} + v_j$ $(v_0 = v_n = 0)$. Thus

$$A_{S_j}(v,v) \le 2[A_{S_j}(v_{j-1},v_{j-1}) + A_{S_j}(v_j,v_j)].$$

The first inequality follows by summing the above inequality from 1 to n.

We now prove the upper estimate for $(\mathfrak{J}_h^{-1}v,v)$. In the bilinear case, note that

$$v_j(x,y_j) = v(x,y_j) \quad \text{and} \quad v_j(x,y_{j\pm1}) = 0.$$

A simple calculation shows that,

$$\begin{aligned}\int_{S_j} &(|D_x v_{j-1}|^2 + |D_x v_j|^2)\,\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{3}\sum_{i=1}^{n-1}[|v(x_i,y_{j-1}) - v(x_{i-1},y_{j-1})|^2 + |v(x_i,y_j) - v(x_{i-1},y_j)|^2] \\ &\le 2\int_{S_j} |v_x|^2\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Consequently,

$$\int_{S_j} a(x,y)(|D_x v_{j-1}|^2 + |D_x v_j|^2)\,\mathrm{d}x\mathrm{d}y \le 2\frac{a_{\max}}{a_{\min}}\int_{S_j} a(x,y)|v_x|^2\,\mathrm{d}x\mathrm{d}y. \tag{5.4}$$

On the other hand, for $(x,y) \in S_j$,

$$|D_y v_{j-1}(x,y)|^2 + |D_y v_j(x,y)|^2 = \frac{1}{h^2}(|v(x,y_{j-1})|^2 + |v(x,y_j)|^2).$$

Since $v_{yy}(x,\theta) \equiv 0$ for θ between y_{j-1} and y_j,

$$\begin{aligned}|v(x,&y_{j-1})|^2 + |v(x,y_j)|^2 \\ &= |v(x,y) + v_y(x,y)(y_{j-1}-y)|^2 + |v(x,y) + v_y(x,y)(y_j - y)|^2 \\ &\le 4|v(x,y)|^2 + 2h^2|v_y(x,y)|^2,\end{aligned}$$

for all $(x,y) \in S_j$. Hence

$$\int_{S_j} b(|D_y v_{j-1}|^2 + |D_y v_j|^2)\,\mathrm{d}x\mathrm{d}y \le \frac{4}{h^2}\int_{S_j} b|v|^2\,\mathrm{d}x\mathrm{d}y + 2\int_{S_j} b|v_y|^2\,\mathrm{d}x\mathrm{d}y. \tag{5.5}$$

Combining (5.4) and (5.5)

$$\begin{aligned} &A_{S_j}(v_{j-1}, v_{j-1}) + A_{S_j}(v_j, v_j) \\ &\qquad \le \frac{4}{h^2} \int_{S_j} bv^2 \,\mathrm{d}x\mathrm{d}y + 2\frac{a_{\max}}{a_{\min}} \int_{S_j} (a|v_x|^2 + b|v_y|^2) \,\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Summing from 1 to n,

$$\begin{aligned} (\mathfrak{I}_h^{-1} v, v) &\equiv \sum_{j=1}^{n} [A_{S_j}(v_{j-1}, v_{j-1}) + A_{S_j}(v_j, v_j)] \\ &\le \left[2\frac{a_{\max}}{a_{\min}}(Av, v) + \frac{4}{h^2}(bv, v) \right]. \end{aligned}$$

This proves the second inequality for the bilinear case.

The proof for the case of linear elements is similar. We write $\tau = \tau^+ \cup \tau^-$. Then

$$D_x v_{j-1} = 0,\ D_x v_j = D_x v \text{ on } \tau^+ \quad \text{and} \quad D_x v_j = 0,\ D_x v_{j-1} = D_x v \text{ on } \tau^-.$$

Therefore

$$\int_\tau a(|D_x v_{j-1}|^2 + |D_x v_j|^2) \,\mathrm{d}x\mathrm{d}y = \int_\tau a|D_x v|^2 \,\mathrm{d}x\mathrm{d}y.$$

On the other hand,

$$|D_y v_{j-1}|^2 + |D_y v_j|^2 = \frac{1}{h^2}(|v(x_{i-1}, y_{j-1})|^2 + |v(x_{i-1}, y_j)|^2) \quad \text{in } \tau^+$$

and

$$|D_y v_{j-1}|^2 + |D_y v_j|^2 = \frac{1}{h^2}(|v(x_i, y_{j-1})|^2 + |v(x_i, y_j)|^2) \qquad \text{in } \tau^-.$$

Since all the second derivatives of v vanish on $\tau^\pm$, we have

$$\begin{aligned} &|v(x_{i-1}, y_{j-1})|^2 + |v(x_{i-1}, y_j)|^2 \\ &\qquad = |v(x,y) + \nabla v(x,y) \cdot (x_{i-1} - x, y_{j-1} - y)|^2 \\ &\qquad\ + |v(x,y) + \nabla v(x,y) \cdot (x_{i-1} - x, y_j - y)|^2 \\ &\qquad \le 4|v(x,y)|^2 + 6h^2 |\nabla v(x,y)|^2, \quad \text{for all} \quad (x,y) \in \tau^+. \end{aligned}$$

A similar estimate holds for $|v(x_i, y_{j-1})|^2 + |v(x_i, y_j)|^2$. Hence,

$$\begin{aligned} &\int_\tau b(|D_y v_{j-1}|^2 + |D_y v_j|^2) \\ &\qquad = \frac{1}{h^2} \left[\int_{\tau^+} b(|v(x_{i-1}, y_{j-1})|^2 + |v(x_{i-1}, y_j)|^2) \,\mathrm{d}x\mathrm{d}y \right. \\ &\qquad\qquad \left. + \int_{\tau^-} b(|v(x_i, y_{j-1})|^2 + |v(x_i, y_j)|^2) \,\mathrm{d}x\mathrm{d}y \right] \\ &\qquad \le \frac{4}{h^2} \int_\tau bv^2 \,\mathrm{d}x\mathrm{d}y + 6 \int b|\nabla v|^2 \,\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Combining the estimates for $\int_\tau a(|D_x v_{j-1}|^2 + |D_x v_j|^2)$ and $\int_\tau b(|D_y v_{j-1}|^2 + |D_y v_j|^2)$ we obtain

$$A_{S_j}(v_{j-1}, v_{j-1}) + A_{S_j}(v_j, v_j) \le \frac{4}{h^2}\int_{S_j} bv^2 \,\mathrm{d}x\mathrm{d}y + \int_{S_j}[(1+6b)v_x^2 + 6bv_y^2]\,\mathrm{d}x\mathrm{d}y.$$

The rest of the proof is identical to that for the bilinear elements. □

We formulate the smoothing property of the line Gauss-Seidel in the next lemma.

Lemma 5.2 *Let M_h consist of piecewise linear or bilinear functions and let $\mathfrak{G}_h$ be the line Gauss-Seidel smoother defined in* (5.2). *Then there is a constant C_2 such that*

$$A(v,v) \le (\bar{\mathfrak{G}}_h^{-1} v, v) \le C_2\left[A(v,v) + \frac{1}{h^2}(bv,v)\right], \quad \textit{for all} \quad v \in M_h,$$

where $\bar{\mathfrak{G}}_h \equiv \mathfrak{G}_h + \mathfrak{G}_h^t - \mathfrak{G}_h^t A_h \mathfrak{G}_h$ is the symmetric line Gauss-Seidel smoother.

Proof. The lower estimate is trivial. The upper estimate follows from the following inequality

$$A(\mathfrak{J}_h A_h v, v) \le C A(\bar{\mathfrak{G}}_h A_h v, v)$$

and the upper estimate for the line Jacobi smoother in Lemma 5.1. □

Remark 5.1 In the proof of Lemma 5.1, we did not make use of (1.4). With minor modifications, we can prove that the results in Lemmas 5.1 and 5.2 hold for general polynomial elements, provided that (1.4) holds. □

6 Multigrid convergence estimate

We now establish a uniform convergence result for the V-cycle multigrid Algorithm 2.1. For simplicity, we only consider linear and bilinear elements. We introduce an initial triangulation $\mathcal{T}_1$ of Ω by partitioning Ω into four smaller equal squares. For linear elements, each square is further decomposed into two triangles by linking the lower-left and upper-right vertices. Let $\{\mathcal{T}_k\}$ be a family of triangulations of Ω, where $\mathcal{T}_k$ is obtained from $\mathcal{T}_{k-1}$ by a halving strategy. Let $\{M_k\}$ be the corresponding family of linear or bilinear finite element spaces defined with respect to $\{\mathcal{T}_k\}$. Denote by $\mathfrak{J}_{h_k}$ and $\mathfrak{G}_{h_k}$ respectively the line Jacobi and the line Gauss-Seidel operators on M_k.

Theorem 6.1 *Let $R_k = \frac{1}{2}\mathfrak{J}_{h_k}$ or $R_k = \mathfrak{G}_{h_k}$. Then there is a positive number δ with $\delta < 1$ independent of k such that the multigrid algorithm defined in Algorithm* 2.1 *satisfies*

$$0 \le A((I - B_k A_k)v, v) \le \delta A(v,v), \quad \textit{for all} \quad v \in M_k.$$

Proof. We first show the result for $R_k = \frac{1}{2}\mathfrak{J}_{h_k}$. In view of Lemma 2.1, we only need to prove that (2.4) holds for $R_k = \frac{1}{2}\mathfrak{J}_{h_k}$. It follows from Lemma 4.3 that

$$\frac{1}{h^2}(b(x,y)(I-P_{k-1})v,(I-P_{k-1})v) \le C_1 A((I-P_{k-1})v,(I-P_{k-1})v).$$

By Lemma 5.1, the weighted line Jacobi smoother, R_k, satisfies

$$(R_k^{-1}\phi,\phi) \le C_2\left[(A_k\phi,\phi)+\frac{1}{h_k^2}(b(x,y)\phi,\phi)\right], \quad \text{for all} \quad \phi \in M_k.$$

Applying the smoothing property of R_k to $\phi = (I-P_{k-1})v$ and using the approximation property of P_{k-1}, we obtain, with $C_M = C_2(1+C_1)$,

$$(R_k^{-1}(I-P_{k-1})v,(I-P_{k-1})v) \le C_M\, A((I-P_{k-1})v,(I-P_{k-1})v).$$

We have thus proved (2.4) and hence the theorem for the case $R_k = \frac{1}{2}\mathfrak{J}_{h_k}$.

The proof for the case $R_k = \mathfrak{G}_{h_k}$ is analogous. We use Lemma 2.2 in the place of Lemma 2.1 and use the smoothing property of the line Gauss-Seidel in Lemma 5.2 instead of Lemma 5.1. □

Since B_kA_k is symmetric in the energy inner product $A(\cdot,\cdot)$, Theorem 6.1 implies that $\|(I-B_kA_k)v\|_A \le \delta\|v\|_A$ for all $v \in M_k$. Consequently, the error operator of the multigrid iteration (2.1) is a uniform contraction in the $\|\cdot\|_A$ norm and the iterates defined in (2.1) satisfy

$$\|u^m - u\|_A \le \delta^m \|u^m - u\|_A.$$

Remark 6.1 It is desirable to avoid the condition in (1.4) in our theory. However it is not clear that this is possible even in the case when $b(x,y)$ is uniformly bounded from above and below. □

Remark 6.2 Note that if $b(x,y) \approx h_k^2$, then the error operators, $I - \frac{1}{2}\mathfrak{J}_kA_k$ and $I-\mathfrak{G}_k$, corresponding to the line Jacobi and the line Gauss-Seidel methods are already uniform contractions on M_k. □

Remark 6.3 Our analysis remains valid for other polynomial finite elements as well. The approximation property is a consequence of the fact that the linear elements are a subspace of these higher order elements and the smoothing property can be established in a way similar to that of the linear element. □

7 Matrix-vector implementation

We now discuss briefly the implementation of the multigrid algorithm using a matrix-vector notation. Let $\{\phi_k^i\}$, $i = 1,\ldots,N_k$, be the nodal basis of M_k. Then each function v in M_k is associated with two vectors, its "coefficient vector" $\tilde{v}$ and its "dual vector" $\underset{\sim}{v}$. The components of the coefficient vector $\tilde{v}$ consist of the coefficients of v with respect to the basis $\{\phi_k^i\}$ and the dual vector, defined by $\underset{\sim}{v} = [(v,\phi_k^i)]$, represents the action of v on the basis $\{\phi_k^i\}$. Corresponding to the operator A_k is the stiffness matrix $\underset{\approx}{A}_k = [A(\phi_k^i, \phi_k^j)]$. Using this notation, the finite element equation $A_k u_k = f_k$ can be written as a linear system of equations:

$$\underset{\approx}{A}_k \tilde{u}_k = \underset{\sim}{f}_k.$$

To define the multigrid algorithm in terms of vectors and matrices, we introduce, for each smoother R_k, a smoothing matrix $\overset{\approx}{R}_k$ satisfying

$$\widetilde{R_k f_k} = \overset{\approx}{R}_k \underset{\sim}{f}_k, \quad \text{for all} \quad f_k \in M_k. \tag{7.1}$$

It is easy to check that the smoothing matrices corresponding to $R_k = \frac{1}{2}\mathfrak{J}_{h_k}$ and $R_k = \mathfrak{G}_{h_k}$ are just the block diagonal and lower block triangular parts of $\underset{\approx}{A}_k$.

Since $M_{k-1} \subset M_k$, the basis functions of M_{k-1} can be expressed in terms of those of M_k, i.e.,

$$\phi_{k-1}^i = \sum_{j=1}^{N_k} \alpha_{ij}^k \phi_k^j.$$

The "interpolation matrix" and the "prolongation matrix" are given respectively by $\Pi_{k-1} = [\alpha_{ij}^k]$ and Π_{k-1}^t.

With the above notation, the coefficient vectors, $\tilde{u}^m$, of the multigrid iterates u^m defined in (2.1) can be computed by

$$\tilde{u}^{m+1} = \tilde{u}^m + \overset{\approx}{B}_k(\underset{\sim}{f}_k - \underset{\approx}{A}_k \tilde{u}^m), \quad m = 0, 1, \dots, \tag{7.2}$$

where $\overset{\approx}{B}_k$ is defined recursively by the following algorithm:

Algorithm 7.1 (Matrix-vector form) *Set* $\overset{\approx}{B}_1 = \underset{\approx}{A}_1^{-1}$. *Assume that* $\overset{\approx}{B}_{k-1}$ *has been defined. Define* $\overset{\approx}{B}_k : R^{N_k} \to R^{N_k}$ *as follows.*

(1) *Pre-smoothing: Set* $\tilde{v}' = \tilde{v} + \overset{\approx}{R}_k^t (\underset{\sim}{f}_k - \underset{\approx}{A}_k \tilde{v})$.

(2) *Correction: Define* $\tilde{v}'' = \tilde{v}' + \Pi_{k-1}^t \tilde{q}$ $(\tilde{v} = 0)$ *where*

$$\tilde{q} = \overset{\approx}{B}_{k-1} \Pi_{k-1}(\underset{\sim}{f}_k - \underset{\approx}{A}_k \tilde{v}').$$

(3) *Post-smoothing: Set* $\overset{\approx}{B}_k \underset{\sim}{f}_k = \tilde{v}'' + \overset{\approx}{R}_k(\underset{\sim}{f}_k - \underset{\approx}{A}_k \tilde{v}'')$.

It is straightforward to check that if $\overset{\approx}{R}_k$ and R_k are related by (7.1), then

$$\widetilde{B_k f_k} = \overset{\approx}{B}_k \underset{\sim}{f}_k, \quad \text{for all} \quad \underset{\sim}{f}_k \in M_k.$$

Therefore, Algorithms 7.1 and 2.1 are the same and $\tilde{u}^m$ defined by (7.2) is the coefficient vector of u^m defined by (2.1), provided that $\underset{\sim}{f}_k$ is the dual vector of f_k and $\tilde{u}^0$ is chosen to be the coefficient vector of u^0.

Bibliographical notes.

Some earlier work on multilevel methods for anisotropic problems can be found in Hackbusch [7] and the references therein.

The multigrid algorithm considered in this section is well known and dates back at least to the early eighties. It is also known that the method performs quite well; cf. Hackbusch [7]. To our knowledge however there is no rigorous proof for the efficiency of the algorithm. Our result can be considered as a generalization of the work of Braess and Hackbusch [1] to the case of anisotropic equations.

Some recent work on multilevel methods for anisotropic problems can be found, for example, in Wittum [14, 15], Stevenson [10, 9, 11, 13, 12], Griebel and Oswald [6], and Hemker [8].

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