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ANALYSIS OF A FINITE PML APPROXIMATION FOR THE THREE DIMENSIONAL TIME-HARMONIC MAXWELL AND ACOUSTIC SCATTERING PROBLEMS

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Abstract. We consider the approximation of the frequency domain three dimensional Maxwell scattering problem using a truncated domain perfectly matched layer (PML) (cf. [3] and [4]). We also treat the time-harmonic PML approximation to the acoustic scattering problem. Following [11], a transitional layer based on spherical geometry is defined which results in a constant coefficient problem outside the transition. A truncated (computational) domain is then defined which covers the transition region. The truncated domain need only have a minimally smooth outer boundary (e.g., Lipschitz continuous). We consider the truncated PML problem which results when a perfectly conducting boundary condition is imposed on the outer boundary of the truncated domain. The existence and uniqueness of solutions to the truncated PML problem will be shown provided that the truncated domain is sufficiently large, e.g., contains a sphere of radius R_t. We also show exponential (in the parameter R_t) convergence of the truncated PML solution to the solution of the original scattering problem inside the transition layer.

Our results are important in that they are the first which show that the truncated PML problem can be posed on a domain with non-smooth outer boundary. This allows the use of approximation based on polygonal meshes. In addition, even though the transition coefficients depend on spherical geometry, they can be made arbitrarily smooth and hence the resulting problems are amenable to numerical quadrature. Approximation schemes based on our analysis are the focus of future research.

1. Introduction

In this paper, we consider the acoustic and electromagnetic scattering problem in three spatial dimensions. Simulations involving these problems are inherently difficult for a number of reasons. First, although the problems are symmetric, they are indefinite. Second, the problems have a scale related to the wavenumber k and so standard discretizations require mesh sizes proportional to k^{-1}. Third, the problems are posed on infinite domains.

The focus of this paper is on the third issue above, i.e., how to deal with the boundary condition at infinity in a computationally effective way. Specifically, we shall study perfectly matched layer (PML) approximations to acoustic and electromagnetic problems. The goal is to demonstrate both the solvability of the continuous PML approximations

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and the convergence of the resulting solutions to the solutions of the original acoustic/electromagnetic problem.

Recently, there has been intensive computational and theoretical research toward understanding the properties of PML approximations. The research into the computational aspects of these methods is the subject of many papers in the engineering literature and we shall not attempt to discuss them here. There is evidence to suggest that this approach is very competitive with standard techniques for computational domain truncation. In the paper by Petropoulos [15] he refers to recent numerical results in [10] which "indicate our reflectionless sponge layer provides levels of numerical reflection from $\partial\Omega^c$ that are comparable to those obtained with the exact ABC [9] for vector spherical waves scattering from a dielectric sphere but at a substantial savings in computational cost."

The original PML method was suggested by Bérenger in [4] and [3]. The observation that a PML method could be considered as a complex change of variable was made by Chew and Weedon [5]. Using this technique, Collino and Monk [6] derived PML equations based on rectangular and polar coordinates. There, they also showed the existence and uniqueness of solutions of the truncated acoustic PML except for a countable number of wave numbers. The formulation of PML equations for (2.1) in spherical coordinates can be found in [13]. Lassas and Sommersalo [11] proved the existence and uniqueness of the PML acoustic approximation on a truncated domain where the outer boundary was circular. In a later paper [12], they extended these results to smooth convex domains in $\mathbb{R}^n$.

To date, there has been relatively little analysis of the truncated electromagnetic PML equations. Techniques for the acoustic problem do not carry over directly to the electromagnetic problem. This stems from the fact that the acoustic problem is strongly elliptic (up to perturbation) while the electromagnetic operator has an infinite dimensional kernel consisting of functions which are gradients. For example, Collino and Monk [6] use a perturbation analysis to derive their existence result. Carrying this argument over to the electromagnetic PML poses significant analytical difficulties requiring the analysis of vector decompositions involving the complex-valued PML coefficient. Recently, Bao and Wu gave a convergence analysis for the electromagnetic scattering problem where the PML layer was a spherical shell [2].

We will present a new analytical approach for the study of the electromagnetic PML equation in this paper. Not only do we derive existence and uniqueness for the PML approximations but our analysis leads to inf-sup conditions on the truncated domain, an essential part of any finite element analysis.

Let Ω (the scatterer) be a domain in $\mathbb{R}^3$. We shall first consider the acoustic scattering problem with a *sound-soft* obstacle. This involves a scalar function u defined on Ω^c, the complement of $\bar{\Omega}$, satisfying

$$\begin{aligned} \Delta u + k^2 u &= 0 \text{ in } \Omega^c, \\ u &= g \text{ on } \partial\Omega, \\ \lim_{\rho\to\infty} \rho(\nabla u \cdot \hat{\boldsymbol{x}} - iku) &= 0. \end{aligned} \tag{1.1}$$

Here $\rho = |\boldsymbol{x}|$, $\hat{\boldsymbol{x}} = \boldsymbol{x}/\rho$, and k is a real positive constant. We have absorbed the medium properties into the constant k.

We will also consider the time-harmonic electromagnetic scattering problem. In this case, we shall assume, for convenience, that Ω is simply connected with only one boundary component. We seek vector fields $\boldsymbol{E}$ and $\boldsymbol{H}$ defined on Ω^c satisfying

$$\begin{aligned} -ik\mu\boldsymbol{H} + \boldsymbol{\nabla}\times\boldsymbol{E} &= \boldsymbol{0}, \text{ in } \Omega^c \\ -ik\epsilon\boldsymbol{E} - \boldsymbol{\nabla}\times\boldsymbol{H} &= \boldsymbol{0}, \text{ in } \Omega^c \\ \boldsymbol{n}\times\boldsymbol{E} &= \boldsymbol{n}\times\boldsymbol{g}, \text{ on } \partial\Omega, \\ \lim_{\rho\to\infty}\rho(\mu\boldsymbol{H}\times\hat{\boldsymbol{x}} - \boldsymbol{E}) &= \boldsymbol{0}. \end{aligned} \tag{1.2}$$

Here $\boldsymbol{g}$ results from a given incidence field, μ is the magnetic permeability, ϵ is the electric permittivity, and $\boldsymbol{n}$ is the outward unit normal on $\partial\Omega$. The last line corresponds to the Silver-Müller condition at infinity. We assume that the coefficients μ and ϵ are real valued, bounded away from zero and constant outside of some ball.

We introduce some notation that will be used in the remainder of the paper. For a domain D, let $L^2(D)$ be the space of (complex valued) square integrable functions on D and $\boldsymbol{L}^2(D) = (L^2(D))^3$ be the space of vector valued L^2-functions. We shall use $(\cdot,\cdot)_\Omega$ to denote the (vector or scalar Hermitian) $L^2(\Omega)$ inner product and $<\cdot,\cdot>_\Gamma$ to denote the (vector or scalar Hermitian) $L^2(\Gamma)$ boundary inner product. When the inner product is on all of $\mathbb{R}^3$, we will use the notation $(\cdot,\cdot)$. The scalar and vector Sobolev spaces on D will be denoted $H^s(D)$ and $\boldsymbol{H}^s(D)$ respectively. Let $\boldsymbol{H}(\mathbf{curl};D)$ be the set of vector valued functions, which along with their curls, are in $\boldsymbol{L}^2(D)$. $\boldsymbol{H}_0(\mathbf{curl};D)$ denotes the functions $\boldsymbol{f}$ in $\boldsymbol{H}(\mathbf{curl};D)$ satisfying $\boldsymbol{n}\times\boldsymbol{f} = \boldsymbol{0}$ on $\partial\Omega$. We assume that $\boldsymbol{n}\times\boldsymbol{g}$ above is the trace $\boldsymbol{n}\times\widehat{\boldsymbol{g}}$ of a function $\widehat{\boldsymbol{g}} \in \boldsymbol{H}(\mathbf{curl};\Omega^c)$ supported close to $\partial\Omega$.

For a subdomain $D \subset \Omega^c$, by extension by zero, we identify $H_0^1(D)$ (respectively, $\boldsymbol{H}_0(\mathbf{curl};D)$) with $\{v \in H_0^1(\Omega^c)$ (respectively, $\boldsymbol{H}_0(\mathbf{curl};\Omega^c)$) $:$ $\mathrm{supp}(v) \subseteq \bar{D}\}$.

2. The Bérenger layer

For convenience, we shall take $\mu = \epsilon = 1$ in (1.2) as all of our results extend to the more general case as long as the coefficients are constant outside of a ball of radius r_0. We can reduce to a single equation involving $\boldsymbol{E}$ by eliminating $\boldsymbol{H}$ in (1.2). This gives

$$\begin{aligned} -\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{E} + k^2\boldsymbol{E} &= \boldsymbol{0} \text{ in } \Omega^c, \\ \boldsymbol{n}\times\boldsymbol{E} &= \boldsymbol{n}\times\boldsymbol{g} \text{ on } \partial\Omega, \\ \lim_{\rho\to\infty}\rho((\boldsymbol{\nabla}\times\boldsymbol{E})\times\hat{\boldsymbol{x}} - ik\boldsymbol{E}) &= \boldsymbol{0}. \end{aligned} \tag{2.1}$$

Throughout this paper, we shall use a sequence of finite subdomains of Ω^c with spherical outer boundaries. Let $r_0 < r_1 < \ldots < r_4$ be an increasing sequence of real numbers and let Ω_i denote (interior of) the open ball B_i of radius r_i excluding $\bar{\Omega}$ (we assume that r_0 is large enough so that the corresponding ball contains $\bar{\Omega}$ and that the origin is contained in Ω). We denote the outer boundary of Ω_i by Γ_i. The values of $r_0, r_1, \ldots, r_4$ are independent of the computational outer boundary scaling parameter R_t (introduced below).

As discussed in [6], the PML problem can be viewed as a complex coordinate transformation. Following [11], a transitional layer based on spherical geometry is defined which results in a constant coefficient problem outside the transition. Given σ_0, r_1, and

r_2, we start with a function $\widetilde{\sigma} \in C^2(\mathbb{R}^+)$ satisfying

$$\begin{aligned} &\widetilde{\sigma}(\rho) = 0 \quad \text{for } 0 \le \rho \le r_1, \\ &\widetilde{\sigma}(\rho) = \sigma_0 \quad \text{for } \rho \ge r_2, \\ &\widetilde{\sigma}(\rho) \text{ increasing for } \rho \in (r_1, r_2). \end{aligned}$$

We define

$$\tilde{\rho} = \rho(1 + i\widetilde{\sigma}) \equiv \rho\widetilde{d}.$$

One obvious construction of such a function $\widetilde{\sigma}$ in the transition layer $r_1 \le \rho \le r_2$ with the above properties is given by the fifth order polynomial,

$$\widetilde{\sigma}(\rho) = \sigma_0 \bigg(\int_{r_1}^{\rho} (t - r_1)^2 (r_2 - t)^2 \, dt \bigg) \bigg(\int_{r_1}^{r_2} (t - r_1)^2 (r_2 - t)^2 \, dt \bigg)^{-1} \quad \text{for } r_1 \le \rho \le r_2.$$

A smoother $\widetilde{\sigma}$ can be constructed by increasing the exponents in the above formula.

Each component of the solution $\boldsymbol{E}$ of (2.1) satisfies the Helmholtz equation with Sommerfeld radiation condition, i.e.,

$$\begin{aligned} \Delta u + k^2 u = 0 \quad \text{for } \rho > r_0, \\ \lim_{\rho \to \infty} \rho(\nabla u \cdot \hat{\boldsymbol{x}} - iku) = 0. \end{aligned} \tag{2.2}$$

Of course, this also holds for the acoustic problem (1.1). It follows that the solution of (2.1) can be expanded

$$\boldsymbol{E} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \boldsymbol{a}_{n,m} h_n^1(k\rho) Y_n^m(\theta, \phi), \quad \text{for } \rho \ge r_0. \tag{2.3}$$

Here $h_n^1(r)$ are spherical Bessel functions of the third kind (Hankel functions), Y_n^m are spherical harmonics (see, e.g., [13] for details) and $\boldsymbol{a}_{n,m}$ are vector valued constants. The solution of the acoustic scattering problem satisfies (2.3) as well with $\boldsymbol{E}$ replaced by u and the vector coefficients $\{\boldsymbol{a}_{n,m}\}$ replaced by scalar coefficients $\{a_{n,m}\}$.

The PML solution in either case is developed in a similar fashion. We illustrate the development in the case of Maxwell's equations. The (infinite domain) PML solution is defined by

$$\widetilde{\boldsymbol{E}} = \begin{cases} \boldsymbol{E}(\boldsymbol{x}) \quad \text{for } |\boldsymbol{x}| \le r_1, \\ \displaystyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \boldsymbol{a}_{n,m} h_n^1(k\tilde{\rho}) Y_n^m(\theta, \phi), \quad \text{for } \rho = |\boldsymbol{x}| \ge r_1. \end{cases}$$

By construction $\widetilde{\boldsymbol{E}}$ and $\boldsymbol{E}$ coincide on Ω_1. Furthermore, the complex shift in the argument of h_n^1 above guarantees exponential decay of $\widetilde{\boldsymbol{E}}$.

The PML solution defined above satisfies a differential equation involving $\tilde{\rho}$ and $\frac{d\tilde{\rho}}{d\rho}$. A simple computation shows that

$$\frac{d\tilde{\rho}}{d\rho} = (1 + i\sigma(\rho)) \equiv d$$

where

$$\sigma(\rho) = \widetilde{\sigma}(\rho) + \rho\widetilde{\sigma}'(\rho).$$

It follows that σ is in $C^1(\mathbb{R}^+)$ and satisfies

$$\begin{aligned}
&\sigma(\rho) = 0 \quad \text{for } 0 \le \rho \le r_1,\\
&\sigma(\rho) > \widetilde{\sigma}(\rho) \quad \text{for } \rho \in (r_1 r_2),\\
&\sigma(\rho) = \sigma_0 \quad \text{for } \rho \ge r_2
\end{aligned}$$

The solution $\widetilde{\boldsymbol{E}}$ satisfies Maxwell's equations using the spherical coordinates $(\tilde{\rho}, \theta, \phi)$ [13]. More precisely,

$$\begin{aligned}
-\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{E}} + k^2 \widetilde{\boldsymbol{E}} &= \boldsymbol{0} \text{ in } \Omega^c,\\
\boldsymbol{n} \times \widetilde{\boldsymbol{E}} &= \boldsymbol{n} \times \boldsymbol{g} \text{ on } \partial\Omega,\\
\widetilde{\boldsymbol{E}} &\text{ bounded at } \infty.
\end{aligned} \tag{2.4}$$

For $\widetilde{\boldsymbol{E}}$ expanded in spherical coordinates,

$$\widetilde{\boldsymbol{E}} = \widetilde{\boldsymbol{E}}_\rho \boldsymbol{e}_\rho + \widetilde{\boldsymbol{E}}_\theta \boldsymbol{e}_\theta + \widetilde{\boldsymbol{E}}_\phi \boldsymbol{e}_\phi,$$

we have

$$\begin{aligned}
\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{E}} = {} & \frac{1}{\widetilde{d}\rho \sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\, \widetilde{\boldsymbol{E}}_\phi) - \frac{\partial \widetilde{\boldsymbol{E}}_\theta}{\partial\phi}\right)\boldsymbol{e}_\rho\\
& + \frac{1}{\rho\widetilde{d}}\left(\frac{1}{\sin\theta}\frac{\partial \widetilde{\boldsymbol{E}}_\rho}{\partial\phi} - \frac{1}{d}\frac{\partial}{\partial\rho}(\widetilde{d}\rho \widetilde{\boldsymbol{E}}_\phi)\right)\boldsymbol{e}_\theta\\
& + \frac{1}{\widetilde{d}\rho}\left(\frac{1}{d}\frac{\partial}{\partial\rho}(\widetilde{d}\rho\widetilde{\boldsymbol{E}}_\theta) - \frac{\partial \widetilde{\boldsymbol{E}}_\rho}{\partial\theta}\right)\boldsymbol{e}_\phi.
\end{aligned} \tag{2.5}$$

The PML approximation in the acoustic case is given by

$$\begin{aligned}
\widetilde{\Delta}\widetilde{u} + k^2\widetilde{u} &= 0 \text{ in } \Omega^c,\\
\widetilde{u} &= g \text{ on } \partial\Omega,\\
\widetilde{u} &\text{ bounded at } \infty.
\end{aligned} \tag{2.6}$$

In polar coordinates (ρ, θ, ϕ),

$$\widetilde{\Delta} v = \frac{1}{\widetilde{d}^2 d\rho^2}\frac{\partial}{\partial\rho}\left(\frac{\widetilde{d}^2\rho^2}{d}\frac{\partial v}{\partial\rho}\right) + \frac{1}{\widetilde{d}^2\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + \frac{1}{\widetilde{d}^2\rho^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}. \tag{2.7}$$

Since the solutions of (2.4) and (2.6) coincide with those of (2.1) and (1.1), respectively, on Ω_1 while rapidly decaying as ρ tends to infinity, it is natural to truncate to a finite computational domain Ω_∞ and impose convenient boundary conditions on the outer boundary of Ω_∞ (which we denote by Γ_∞). We shall always require that the transitional region is contained in Ω_∞, i.e., $\bar{\Omega}_2 \subset \Omega_\infty$. We introduce the parameter R_t and assume that Ω_∞ contains the sphere of radius R_t. Our analysis will require only a fixed (Lipshitz continuous) outer boundary shape (but one that we enlarge by our dilation parameter R_t). Of course, in practice, it is often convenient to take a polyhedral outer boundary.

It will be critical to keep track of the relation between constants appearing in the inequalities and the scaling parameter R_t. Our constants are independent of R_t and will be denoted generically with the letter C.

The truncated PML approximations are then given as follows. In the case of Maxwell's problem, we consider the truncated PML problem involving a vector function $\widetilde{\boldsymbol{E_t}}$ defined on Ω_∞ and satisfying

$$-\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{E_t}} + k^2 \widetilde{\boldsymbol{E_t}} = \boldsymbol{0} \text{ in } \Omega_\infty,$$
$$\boldsymbol{n} \times \widetilde{\boldsymbol{E_t}} = \boldsymbol{n} \times \boldsymbol{g} \text{ on } \partial\Omega, \tag{2.8}$$
$$\boldsymbol{n} \times \widetilde{\boldsymbol{E_t}} = \boldsymbol{0} \text{ on } \Gamma_\infty.$$

Analogously, for the acoustics problem, we consider $\widetilde{u}_t$ defined on Ω_∞ satisfying

$$\widetilde{\Delta}\widetilde{u}_t + k^2 \widetilde{u}_t = 0 \text{ in } \Omega_\infty,$$
$$\widetilde{u}_t = g \text{ on } \partial\Omega, \tag{2.9}$$
$$\widetilde{u}_t = 0 \text{ on } \Gamma_\infty.$$

Remark 2.1. It is possible to use and analyze other conditions on the outer boundary. We choose Dirichlet conditions for convenience.

3. Analysis of the truncated acoustic PML (2.9)

In this section, we will prove that the truncated PML acoustic problem (2.9) has a unique weak solution which converges exponentially to the solution of (1.1) near the obstacle. We will first prove uniqueness for (2.9). To do this we will use a duality argument. A similar technique was used in [8] for the exterior Helmholtz problem to estimate the effect of truncating the infinite domain and imposing an approximate absorbing boundary condition.

We first consider a weak formulation of (2.6). Define the sesquilinear form,

$$\begin{aligned} b(v,\chi) = k^2(\widetilde{d}^2 v, \chi)_{\Omega^c} &- \left(\frac{\widetilde{d}^2}{d}\frac{\partial v}{\partial \rho}, \frac{\partial}{\partial \rho}\left(\frac{\chi}{\bar{\bar{d}}}\right)\right)_{\Omega^c} \\ &- \left(\frac{1}{\rho^2}\frac{\partial v}{\partial \theta}, \frac{\partial \chi}{\partial \theta}\right)_{\Omega^c} - \left(\frac{1}{\rho^2 \sin^2\theta}\frac{\partial v}{\partial \phi}, \frac{\partial \chi}{\partial \phi}\right)_{\Omega^c}. \end{aligned} \tag{3.1}$$

This form is well defined for $v \in H^1(\Omega^c)$ and $\chi \in H^1(\Omega^c)$ and results from (2.6) and integration by parts. For $g \in H^{1/2}(\partial\Omega)$, let $\hat{g}$ be an $H^1(\Omega^c)$ bounded extension of g supported in Ω_0. The weak solution of (2.6) is the function $\widetilde{u} = \hat{g} - w$ where $w \in H^1_0(\Omega^c)$ satisfies

$$b(w,\phi) = b(\hat{g},\phi) \quad \text{for all } \phi \in H^1_0(\Omega^c). \tag{3.2}$$

We will subsequently show that the variational problem (3.2) is well posed and that $\widetilde{u}$ is well defined and independent of the particular extension $\hat{g}$.

To employ the duality technique, we need to consider the adjoint source problem on the infinite domain. For $\Phi \in L^2(\Omega^c)$, let $\hat{z} \in H^1_0(\Omega^c)$ satisfy

$$b(\chi,\hat{z}) = (\chi,\Phi)_{\Omega^c}, \quad \text{for all } \chi \in H^1_0(\Omega^c). \tag{3.3}$$

It is immediate that $\hat{z} = \bar{d}\bar{z}$ ($\bar{z}$ denotes the complex conjugate of z), where z satisfies

$$b(z,\chi) = (\bar{\Phi}/d, \chi)_{\Omega^c}, \quad \text{for all } \chi \in H^1_0(\Omega^c). \tag{3.4}$$

The above problems are well posed as is shown in the following theorem.

Theorem 3.1. *Let Φ be in $L^2(\Omega^c)$. Problems (3.4) and (3.3) have unique solutions $z, \hat{z} \in H_0^1(\Omega^c)$ satisfying*

$$\|z\|_{H^1(\Omega^c)} \leq C\|\Phi\|_{L^2(\Omega^c)} \text{ and } \|\hat{z}\|_{H^1(\Omega^c)} \leq C\|\Phi\|_{L^2(\Omega^c)}. \tag{3.5}$$

To prove the above theorem and subsequent results, we shall require the following theorem which follows easily from a theorem due to Peetre [14] and Tartar [16] (see, e.g. Theorem 2.1 of [7]).

Theorem 3.2. *Let $A(\cdot,\cdot)$ be bounded sesquilinear form on a complex Hilbert space V with norm $\|\cdot\|_V$. Let W be another Hilbert space and T a compact operator from V to W. Suppose that the only solution of*

$$A(u, v) = 0 \quad \text{for all } v \in V$$

is $u = 0$ and that

$$\|u\|_V \leq C_1 \sup_{v\in V} \frac{|A(u, v)|}{\|v\|_V} + C_2\|Tu\|_W \quad \text{for all } u \in V.$$

Then, there exists $C_3 > 0$ such that for all $u \in V$,

$$\|u\|_V \leq C_3 \sup_{v\in V} \frac{|A(u, v)|}{\|v\|_V}. \tag{3.6}$$

Proof of Theorem 3.1. We will use Theorem 3.2 to show that the form (3.1) satisfies an inf-sup condition on $H_0^1(\Omega^c)$. To this end we break the form into two parts as follows:

$$b(v, \chi) = b_1(v, \chi) + I(v, \chi) \tag{3.7}$$

where

$$b_1(v, \chi) = k^2(d_0^2 u, \chi)_{\Omega^c} - \left(\frac{\widetilde{d}^2}{d^2}\frac{\partial v}{\partial \rho}, \frac{\partial \chi}{\partial \rho}\right)_{\Omega^c} - \left(\frac{1}{\rho^2}\frac{\partial v}{\partial \theta}, \frac{\partial \chi}{\partial \theta}\right)_{\Omega^c} - \left(\frac{1}{\rho^2 \sin^2\theta}\frac{\partial v}{\partial \phi}, \frac{\partial \chi}{\partial \phi}\right)_{\Omega^c}, \tag{3.8}$$

and

$$I(v, \chi) = \left(\frac{\widetilde{d}^2 d'}{d^3}\frac{\partial v}{\partial \rho}, \chi\right)_{\Omega^c} + k^2((\widetilde{d}^2 - d_0^2)v, \chi)_{\Omega^c}, \tag{3.9}$$

where $d_0 = 1 + i\sigma_0$. Notice that d' and $(\widetilde{d}^2 - d_0^2)$ both vanish for $\rho \geq r_2$. Hence

$$|I(v, v)| \leq C\|v\|_{L^2(\Omega_2)}\|v\|_{H^1(\Omega^c)}. \tag{3.10}$$

Recall that $\widetilde{d} = 1 + i\widetilde{\sigma}$ and $d = 1 + i\sigma$ and that $\sigma \geq \widetilde{\sigma}$. It follows easily that there is a positive real number α such that that

$$Re[d_0^2(1 + i\alpha)] \leq -C_1 < 0 \text{ and } Re\left[\frac{\widetilde{d}^2}{d^2}(1 + i\alpha)\right] \geq C_2 > 0, \tag{3.11}$$

for α large enough. In fact, it suffices to choose $\alpha > max[(1 - \sigma_0^2)/2\sigma_0, \sigma_M]$, where σ_M is the maximum of σ. It follows from (3.8) and (3.11) that

$$(1 + \alpha^2)^{1/2}|b_1(v, v)| \geq |Re[(1 + i\alpha)b_1(v, v)]| \geq C\|v\|^2_{H^1(\Omega^c)}. \tag{3.12}$$

It immediately follows from (3.10) and (3.12) that

$$\|v\|_{H^1(\Omega^c)} \le C\bigg(\sup_{w\in H^1(\Omega^c)} \frac{|b(v,w)|}{\|w\|_{H^1(\Omega^c)}} + \|v\|_{L^2(\Omega_2)}\bigg).$$

Now, using the argument in [6], we have the uniqueness property that if $v \in H^1_0(\Omega^c)$ and $b(v,\phi) = 0$ for all $\phi \in C_0^\infty(\Omega^c)$ then $v = 0$. Theorem 3.2 (with $T : H^1(\Omega^c) \to L^2(\Omega_2)$, the imbedding into $L^2(\Omega_2)$ of the restriction of elements of $H^1(\Omega^c)$ to Ω_2) then gives the inf-sup condition

$$\|v\|_{H^1(\Omega^c)} \le C \sup_{\phi\in C_0^\infty(\Omega^c)} \frac{|b(v,\phi)|}{\|\phi\|_{H^1(\Omega^c)}}, \quad \text{for all } v \in H^1_0(\Omega^c). \tag{3.13}$$

The corresponding inf-sup condition for the adjoint problem follows from the identity

$$b(\phi, v) = b(\bar v/d, \bar d\bar\phi). \tag{3.14}$$

Hence, by the generalized Lax-Milgram Lemma, there exists a unique $z \in H^1_0(\Omega^c)$ satisfying (3.4). Moreover,

$$\|z\|_{H^1(\Omega^c)} \le C \sup_{\phi\in C_0^\infty(\Omega^c)} \frac{|b(z,\phi)|}{\|\phi\|_{H^1(\Omega^c)}} \le C\|\Phi\|_{L^2(\Omega^c)}.$$

This completes the proof of the theorem. □

Remark 3.1. Applying a standard lifting estimate, the proof of the above theorem implies that the solution $\widetilde u$ of (2.6) satisfies

$$\|\widetilde u\|_{H^1(\Omega^c)} \le C\|\hat g\|_{H^1(\Omega_0)} \le C\|g\|_{H^{1/2}(\partial\Omega)}.$$

In addition, the inf-sup condition proved above immediately implies that $\widetilde u$ is independent of the choice of extension $\hat g$.

We will first prove uniqueness for (2.9). In order to do this we will need the following two propositions. These propositions will be used extensively throughout the remainder of this paper. The first is a classical interior estimate for the solution of an elliptic equation. The proof is elementary.

Proposition 3.1. *Suppose that w satisfies the Helmholtz equation*

$$\Delta w + \beta w = 0 \tag{3.15}$$

in a domain D with a (possibly complex) constant β. If D_1 is a subdomain, whose closure is contained in D, then

$$\|w\|_{H^2(D_1)} \le C\|w\|_{L^2(D)}. \tag{3.16}$$

We also need the following proposition.

Proposition 3.2. *Assume that w is bounded at infinity and satisfies (3.15) in $\Omega^c \setminus \bar\Omega_2$ with $\beta = k^2 d_0^2$. Set $S_\gamma = \{\boldsymbol{x} : dist(\boldsymbol{x}, \Gamma_\infty) < \gamma\}$ with γ fixed independent of $R_t > r_4$ and small enough that $\bar S_\gamma$ is in $\Omega^c \setminus \bar\Omega_4$. Then*

$$\|w\|_{L^2(S_\gamma)} \le Ce^{-\sigma_0 k R_t}\|w\|_{L^2(\Omega_4)}.$$

Proof. The fundamental solution of (3.15) with $\beta = k^2 d_0^2$ is

$$\psi(\boldsymbol{x},\boldsymbol{y}) = -\frac{\exp(ikd_0|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}.$$

For any point $\boldsymbol{x}$ in S_γ

$$w(\boldsymbol{x}) = \int_{\Gamma_3} w(\boldsymbol{y})\frac{\partial\psi(\boldsymbol{x},\boldsymbol{y})}{\partial r_{\boldsymbol{y}}}\,dS_{\boldsymbol{y}} - \int_{\Gamma_3}\frac{\partial w(\boldsymbol{y})}{\partial r_{\boldsymbol{y}}}\psi(\boldsymbol{x},\boldsymbol{y})\,dS_{\boldsymbol{y}}. \tag{3.17}$$

Note that there is no contribution above from infinity. Indeed, since w is bounded at infinity it can be written as a (scalar) expansion of the form of (2.3) with k replaced by $d_0 k$. In addition, ψ decays rapidly at infinity since d_0 has a positive imaginary part and so the outer boundary contribution limits to zero.

Using Schwarz's inequality and the properties of ψ it is easy to see that

$$|w(\boldsymbol{x})|^2 \le Ce^{-2\sigma_0 kR_t}\left(\|w\|^2_{L^2(\Gamma_3)} + \left\|\frac{\partial w}{\partial r}\right\|^2_{L^2(\Gamma_3)}\right)\left(\int_{\Gamma_3}\frac{dS_{\boldsymbol{y}}}{|\boldsymbol{x}-\boldsymbol{y}|^2}\right). \tag{3.18}$$

Integrating over S_γ, using a standard trace inequality and Proposition 3.1 we obtain Proposition 3.2. $\square$

We can now prove the following theorem.

Theorem 3.3. *Let u be in $H^1_0(\Omega_\infty)$ and satisfy (2.9) with $g=0$. Then, for R_t large enough, $u=0$. That is to say if $u\in H^1_0(\Omega_\infty)$ satisfies $b(u,\psi)=0$ for all $\psi\in H^1_0(\Omega_\infty)$ then $u=0$.*

Proof. Let u be in $H^1_0(\Omega_\infty)$ satisfy (2.9) with $g=0$, Φ be in $L^2(\Omega^c)$ with support in Ω_4 and $\hat z\in H^1_0(\Omega^c)$ be the solution of (3.3). Then

$$(u,\Phi)_{\Omega_4} = b(u,\hat z) = <\frac{\partial u}{\partial \boldsymbol{n}},\hat z>_{\Gamma_\infty} \tag{3.19}$$

Let $\widetilde H^1(\Omega_\infty\setminus\bar\Omega_3)$ denote the set of functions in $H^1(\Omega_\infty\setminus\bar\Omega_3)$ which vanish on Γ_3. Define the norm

$$\|w\|_{H^{-1/2}(\Gamma_\infty)} = \sup_{\phi\in\widetilde H^1(\Omega_\infty\setminus\bar\Omega_3)}\frac{|<w,\phi>_{\Gamma_\infty}|}{\|\phi\|_{H^1(\Omega_\infty\setminus\bar\Omega_3)}}.$$

Let χ be a smooth cutoff function with support $\bar D_1$ in S_γ (of Proposition 3.2) which is one on Γ_∞. Applying Propositions 3.1 and 3.2 to $\hat z$ and (3.5), it follows that

$$\begin{aligned}|(u,\Phi)_{\Omega_4}| &\le C\frac{|<\frac{\partial u}{\partial\boldsymbol{n}},\chi\hat z>_{\Gamma_\infty}|}{\|\chi\hat z\|_{H^1(S_\gamma)}}\|\hat z\|_{H^1(D_1)}\\ &\le C\frac{|<\frac{\partial u}{\partial\boldsymbol{n}},\chi\hat z>_{\Gamma_\infty}|}{\|\chi\hat z\|_{H^1(S_\gamma)|}}\|\hat z\|_{L^2(S_\gamma)}\\ &\le Ce^{-\sigma_0 kR_t}\|\Phi\|_{L^2(\Omega_4)}\left\|\frac{\partial u}{\partial\boldsymbol{n}}\right\|_{H^{-1/2}(\Gamma_\infty)}.\end{aligned} \tag{3.20}$$

We next estimate the negative norm on the right hand side above. Let $\hat h$ be in $\widetilde H^1(\Omega_\infty\setminus\bar\Omega_2)$ and be equal to zero in $\Omega_3\setminus\bar\Omega_2$. Let $\psi\in H^1_0(\Omega_\infty\setminus\bar\Omega_2)$ satisfy

$$(\nabla\psi,\nabla\theta)_{\Omega_\infty\setminus\bar\Omega_2} - k^2\bar d_0^2(\psi,\theta)_{\Omega_\infty\setminus\bar\Omega_2} = (\nabla\hat h,\nabla\theta)_{\Omega_\infty\setminus\bar\Omega_2} - k^2\bar d_0^2(\hat h,\theta)_{\Omega_\infty\setminus\bar\Omega_2}$$

for all $\theta \in H^1_0(\Omega_\infty \setminus \bar\Omega_2)$. This problem is well posed since d_0^2 has a nonzero imaginary part. It follows that

$$\|\psi\|_{H^1(\Omega_\infty\setminus\bar\Omega_2)} \le C\|\hat h\|_{H^1(\Omega_\infty\setminus\bar\Omega_2)}.$$

We set $h = \hat h - \psi$. Note that both u and h satisfy homogeneous equations in $\Omega_\infty \setminus \bar\Omega_2$, i.e.,

$$\Delta u + k^2 d_0^2 u = 0, \qquad \Delta h + k^2 \bar d_0^2 h = 0. \tag{3.21}$$

Now, using Green's identity,

$$< \frac{\partial u}{\partial \boldsymbol{n}}, \hat h >_{\Gamma_\infty} = < \frac{\partial u}{\partial \boldsymbol{n}}, h >_{\Gamma_\infty} = - < \frac{\partial u}{\partial \boldsymbol{n}}, h >_{\Gamma_3} + < u, \frac{\partial h}{\partial \boldsymbol{n}} >_{\Gamma_3} . \tag{3.22}$$

Finally, using Proposition 3.1 (with D_1 a domain containing Γ_3 whose closure is in $\Omega_4 \setminus \bar\Omega_2$),

$$\begin{aligned} | < \frac{\partial u}{\partial \boldsymbol{n}}, h >_{\Gamma_3} - < u, \frac{\partial h}{\partial \boldsymbol{n}} >_{\Gamma_3} | &\le C\|u\|_{H^2(D_1)}\|h\|_{H^2(D_1)} \\ &\le C\|u\|_{L^2(\Omega_4)}\|h\|_{L^2(\Omega_4)} \\ &\le C\|u\|_{L^2(\Omega_4)}\|\hat h\|_{H^1(\Omega_\infty\setminus\bar\Omega_3)}. \end{aligned} \tag{3.23}$$

Combining the above results shows that

$$\left\|\frac{\partial u}{\partial \boldsymbol{n}}\right\|_{H^{-1/2}(\Gamma_\infty)} \le C\|u\|_{L^2(\Omega_4)}$$

and hence, using (3.20), we have

$$\|u\|_{L^2(\Omega_4)} \le Ce^{-\sigma_0 k R_t}\|u\|_{L^2(\Omega_4)}, \tag{3.24}$$

i.e., u vanishes on Ω_4 provided R_t is taken large enough. It follows by unique continuation that u vanishes on all of Ω_∞. This completes the proof of the uniqueness theorem. □

We next give a weak form of Problem (2.9). For $g \in H^{1/2}(\partial\Omega)$ let $\hat g$ be an $H^1(\Omega^c)$ bounded extension of g with support in Ω_0. The weak solution of (2.9) is the function $\widetilde u_t = \hat g - w$ where $w \in H^1_0(\Omega_\infty)$ satisfies

$$b(w,\phi) = b(\hat g,\phi) \quad \text{for all } \phi \in H^1_0(\Omega_\infty). \tag{3.25}$$

The next theorem shows existence and gives error estimates for the weak solution.

Theorem 3.4. *The variational problem (3.25), for R_t sufficiently large, has a unique solution. The resulting weak solution $\widetilde u_t$ of (2.9) is well defined and independent of the extension $\hat g$. Finally,*

$$\|\widetilde u - \widetilde u_t\|_{L^2(\Omega_4)} \le Ce^{-2\sigma_0 k R_t}\|g\|_{H^{1/2}(\partial\Omega)}.$$

Here $\widetilde u$ is the solution of (2.6).

Remark 3.2. The above theorem shows that $\widetilde u_t$ converges exponentially on Ω_4 to $\widetilde u$ as $R_t \to \infty$. It follows that $\widetilde u_t$ converges exponentially to u on Ω_1.

Proof. We note that (3.10) and (3.12) hold on the restricted space $H^1_0(\Omega_\infty)$ so the uniqueness result of the previous theorem and Theorem 3.2 implies the inf-sup condition

$$\|v\|_{H^1(\Omega_\infty)} \le C \sup_{\phi\in C_0^\infty(\Omega_\infty)} \frac{|b(v,\phi)|}{\|\phi\|_{H^1(\Omega_\infty)}}, \quad \text{for all } v \in H^1_0(\Omega_\infty). \tag{3.26}$$

Uniqueness for the adjoint problem on $H_0^1(\Omega_\infty)$ follows from Theorem 3.3 and (3.14). This implies the existence and uniqueness of solutions to (3.25). It is easy to see that the resulting function $\widetilde{u}_t$ is independent of extension $\hat{g}$.

To finish the proof, we need to show that $\widetilde{u}_t$ converges to $\widetilde{u}$, the solution of (2.6), in $L^2(\Omega_4)$ and that the convergence is exponential as R_t increases beyond some threshold. To see this set $\widetilde{e} = \widetilde{u} - \widetilde{u}_t$. As in the proof of Theorem 3.3, let Φ be in $L^2(\Omega^c)$ with support in Ω_4 and $\hat{z} \in H_0^1(\Omega^c)$ be the solution of (3.3). Let $\mathcal{L}$ denote the formal adjoint of the operator $\widetilde{d}^2\widetilde{\Delta}$. Since $\widetilde{\Delta}$ is a multiple of Δ except on the transition layer $r_1 < \rho < r_2$,

$$(3.27)\qquad \begin{aligned}(\widetilde{e}, \Phi)_{\Omega_4} &= -(\widetilde{e}, \mathcal{L}\hat{z})_{\Omega_\infty} = - < \widetilde{u}, \frac{\partial \hat{z}}{\partial \boldsymbol{n}} >_{\Gamma_\infty} + b_\infty(\widetilde{e}, \hat{z}) \\ &= < \frac{\partial \widetilde{e}}{\partial \boldsymbol{n}}, \hat{z} >_{\Gamma_\infty} - < \widetilde{u}, \frac{\partial \hat{z}}{\partial \boldsymbol{n}} >_{\Gamma_\infty} .\end{aligned}$$

Here $b_\infty(\cdot,\cdot)$ denotes the form on $H^1(\Omega_\infty) \times H^1(\Omega_\infty)$ which results from replacing the domain of integration Ω^c in (3.1) by Ω_∞.

To handle the first term on the right hand side of (3.27), we shall use estimates in the proof of Theorem 3.3 (with u replaced by $\widetilde{e}$). As in (3.20),

$$\left| < \frac{\partial \widetilde{e}}{\partial \boldsymbol{n}}, \hat{z} >_{\Gamma_\infty} \right| \le Ce^{-\sigma_0 k R_t} \|\Phi\|_{L^2(\Omega_4)} \left\| \frac{\partial \widetilde{e}}{\partial \boldsymbol{n}} \right\|_{H^{-1/2}(\Gamma_\infty)}.$$

We estimate the negative norm again following the proof of Theorem 3.3 but replace (3.22) with

$$\begin{aligned}< \frac{\partial \widetilde{e}}{\partial \boldsymbol{n}}, \hat{h} >_{\Gamma_\infty} &= < \frac{\partial \widetilde{e}}{\partial \boldsymbol{n}}, h >_{\Gamma_\infty} \\ &= - < \frac{\partial \widetilde{e}}{\partial \boldsymbol{n}}, h >_{\Gamma_3} + < \widetilde{e}, \frac{\partial h}{\partial \boldsymbol{n}} >_{\Gamma_3} + < \widetilde{u}, \frac{\partial h}{\partial \boldsymbol{n}} >_{\Gamma_\infty} .\end{aligned}$$

The first two terms on the right hand side above are estimated exactly as in (3.23). For the last, we note that because h satisfies (3.21),

$$\left\| \frac{\partial h}{\partial \boldsymbol{n}} \right\|_{H^{-1/2}(\Gamma_\infty)} \le C\|h\|_{H^1(\Omega_\infty \setminus \bar{\Omega}_2)}.$$

Thus,

$$\left| < \widetilde{u}, \frac{\partial h}{\partial \boldsymbol{n}} >_{\Gamma_\infty} \right| \le C\|\widetilde{u}\|_{H^1(D_1)} \|\hat{h}\|_{H^1(\Omega_\infty \setminus \bar{\Omega}_2)}$$

where $\bar{D}_1 \subset S_\gamma$. Applying Propositions 3.1 and 3.2 gives

$$\left| < \widetilde{u}, \frac{\partial h}{\partial \boldsymbol{n}} >_{\Gamma_\infty} \right| \le Ce^{-\sigma_0 k R_t} \|\widetilde{u}\|_{L^2(\Omega_4)} \|\hat{h}\|_{H^1(\Omega_\infty \setminus \bar{\Omega}_2)}.$$

Combining the above gives

$$(3.28)\quad |(\widetilde{e}, \Phi)_{\Omega_4}| \le Ce^{-\sigma_0 k R_t} \|\Phi\|_{L^2(\Omega_4)} (\|\widetilde{e}\|_{L^2(\Omega_4)} + e^{-\sigma_0 k R_t} \|\widetilde{u}\|_{L^2(\Omega_4)}) + \left| < \widetilde{u}, \frac{\partial \hat{z}}{\partial \boldsymbol{n}} >_{\Gamma_\infty} \right|.$$

Now, using a standard trace inequality, Theorem 3.1, Proposition 3.1 and Proposition 3.2, we obtain

$$(3.29)\qquad \left| < \widetilde{u}, \frac{\partial \hat{z}}{\partial \boldsymbol{n}} >_{\Gamma_\infty} \right| \le Ce^{-2\sigma_0 k R_t} \|\widetilde{u}\|_{L^2(\Omega_4)} \|\Phi\|_{L^2(\Omega_4)}.$$

Thus we have

$$|(\widetilde{e}, \Phi)_{\Omega_4}| \leq C(e^{-\sigma_0 k R_t}\|\widetilde{e}\|_{L^2(\Omega_4)} + e^{-2\sigma_0 k R_t}\|\widetilde{u}\|_{L^2(\Omega_4)})\|\Phi\|_{L^2(\Omega_4)}.$$

¿From this and Remark 3.1, it follows that

$$\|\widetilde{e}\|_{L^2(\Omega_4)} \leq Ce^{-\sigma_0 k R_t}\|\widetilde{e}\|_{L^2(\Omega_4)} + Ce^{-2\sigma_0 k R_t}\|g\|_{H^{1/2}(\partial\Omega)}.$$

Hence, for R_t large enough, we obtain the convergence estimate

$$\|\widetilde{u} - \widetilde{u}_t\|_{L^2(\Omega_4)} \leq Ce^{-2\sigma_0 k R_t}\|g\|_{H^{1/2}(\partial\Omega)}.$$

This completes the proof of the theorem. □

4. Uniqueness for the truncated electromagnetic PML problem.

Following [13], we define the diagonal matrices (in spherical coordinates)

$$\boldsymbol{A}\boldsymbol{v} = \widetilde{d}^{-2} v_\rho \boldsymbol{e}_\rho + (\widetilde{d}d)^{-1}(v_\theta \boldsymbol{e}_\theta + v_\phi \boldsymbol{e}_\phi)$$

and

$$\boldsymbol{B}\boldsymbol{v} = d v_\rho \boldsymbol{e}_\rho + \widetilde{d}(v_\theta \boldsymbol{e}_\theta + v_\phi \boldsymbol{e}_\phi).$$

Then, $\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{E}} = \boldsymbol{A}\, \boldsymbol{\nabla}\times (\boldsymbol{B}\widetilde{\boldsymbol{E}})$.

We first define a weak form of the PML problem (2.8) by setting $\widetilde{\boldsymbol{E_t}} = \widehat{\boldsymbol{g}} - \boldsymbol{w}$ and setting up a variational problem for $\boldsymbol{\Xi} = \boldsymbol{B}\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega_\infty)$, i.e.,

$$\mathcal{A}(\boldsymbol{\Xi}, \boldsymbol{\Psi}) = \mathcal{A}(\boldsymbol{B}\widehat{\boldsymbol{g}}, \boldsymbol{\Psi}) \quad \text{for all } \boldsymbol{\Psi} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega_\infty). \tag{4.1}$$

Here the sesquilinear form $\mathcal{A}$ is given by

$$\mathcal{A}(\boldsymbol{\Theta}, \boldsymbol{\Psi}) \equiv -({\boldsymbol{\mu}_b}^{-1}\, \boldsymbol{\nabla}\times \boldsymbol{\Theta}, \boldsymbol{\nabla}\times\boldsymbol{\Psi})_{\Omega^c} + k^2(\boldsymbol{\mu}_b\boldsymbol{\Theta}, \boldsymbol{\Psi})_{\Omega^c} \quad \text{for all } \boldsymbol{\Theta}, \boldsymbol{\Psi} \in \boldsymbol{H}(\mathbf{curl}; \Omega^c)$$

and $\boldsymbol{\mu}_b$ is the three by three matrix which corresponds to the diagonal matrix $(\boldsymbol{A}\boldsymbol{B})^{-1}$ in spherical coordinates. As usual, we define the form on the larger space and consider the space $\boldsymbol{H}_0(\mathbf{curl}; \Omega_\infty)$ as the subset defined by extension by zero.

Our first task is to show uniqueness when R_t is sufficiently large. That is if $\boldsymbol{\Theta}$ is in $\boldsymbol{H}_0(\mathbf{curl}; \Omega_\infty)$ and

$$\mathcal{A}(\boldsymbol{\Theta}, \boldsymbol{\Psi}) = 0 \quad \text{for all } \boldsymbol{\Psi} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega_\infty) \tag{4.2}$$

then $\boldsymbol{\Theta} = \mathbf{0}$.

As was done in the analysis of the acoustic problem, we will again use a duality argument. We consider the adjoint source problem: For $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\Omega^c)$, find $\hat{\boldsymbol{z}} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega^c)$ satisfying

$$\mathcal{A}(\boldsymbol{\Theta}, \hat{\boldsymbol{z}}) = (\boldsymbol{\Theta}, \boldsymbol{\Phi})_{\Omega^c} \quad \text{for all } \boldsymbol{\Theta} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega^c). \tag{4.3}$$

It is immediate that $\hat{\boldsymbol{z}} = \bar{\boldsymbol{z}}$, where $\boldsymbol{z}$ is the solution of

$$\mathcal{A}(\boldsymbol{z}, \boldsymbol{\Theta}) = (\bar{\boldsymbol{\Phi}}, \boldsymbol{\Theta})_{\Omega^c} \quad \text{for all } \boldsymbol{\Theta} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega^c). \tag{4.4}$$

We need the following theorem.

Theorem 4.1. *Let $\boldsymbol{\Phi}$ be in $\boldsymbol{L}^2(\Omega^c)$. Problems (4.4) and (4.3) have unique solutions $\boldsymbol{z}, \hat{\boldsymbol{z}} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega^c)$ satisfying*

$$\|\boldsymbol{z}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)} = \|\hat{\boldsymbol{z}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)} \leq C\|\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\Omega^c)}. \tag{4.5}$$

For the proof of this theorem, we require the following lemma whose proof appears in the appendix.

Lemma 4.1. *Let D either be Ω_∞ or Ω^c, respectively. Set*

$$\boldsymbol{X}(D) = \boldsymbol{H}_0(\mathbf{curl}; D) \cap \boldsymbol{H}^0(\mathrm{div}; \boldsymbol{\mu}_{\mathrm{b}}, D),$$

where $\boldsymbol{H}^0(\mathrm{div}; \boldsymbol{\mu}_{\mathrm{b}}, D) = \{\boldsymbol{U} \in \boldsymbol{L}^2(D) \ : \ \nabla \cdot (\boldsymbol{\mu}_{\mathrm{b}} \boldsymbol{U}) = 0\}$*. Then the functions in $\boldsymbol{X}(D)$, restricted to ω, are in $\boldsymbol{H}^s(\omega)$ for some $s > 1/2$ where $\omega = \Omega_\infty$ or $\omega = \Omega_2$, respectively.*

Proof of Theorem 4.1. We will prove that, for $\boldsymbol{W} \in \boldsymbol{X}(\Omega^c)$,

$$\|\boldsymbol{W}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)} \le C \sup_{\boldsymbol{V}\in \boldsymbol{X}(\Omega^c)} \frac{|\mathcal{A}(\boldsymbol{W}, \boldsymbol{V})|}{\|\boldsymbol{V}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)}}. \tag{4.6}$$

To this end, set $\mathcal{B}(\boldsymbol{U}, \boldsymbol{V}) = \mathcal{A}(\boldsymbol{U}, \bar{\eta}\boldsymbol{V})$, where $\eta = \widetilde{d}^2/d$. Then

$$\begin{aligned} \mathcal{B}(\boldsymbol{U}, \boldsymbol{V}) &= -({\boldsymbol{\mu}_b}^{-1}\, \boldsymbol{\nabla}\times \boldsymbol{U}, \boldsymbol{\nabla}\times\bar{\eta}\boldsymbol{V})_{\Omega^c} + k^2(\eta\boldsymbol{\mu}_b\boldsymbol{U}, \boldsymbol{V})_{\Omega^c} \\ &= -(\eta{\boldsymbol{\mu}_b}^{-1}\, \boldsymbol{\nabla}\times \boldsymbol{U}, \boldsymbol{\nabla}\times\boldsymbol{V})_{\Omega^c} - ({\boldsymbol{\mu}_b}^{-1}\, \boldsymbol{\nabla}\times \boldsymbol{U}, (\nabla\bar{\eta}) \times \boldsymbol{V})_{\Omega^c} \\ &\qquad + k^2(\eta\boldsymbol{\mu}_b\boldsymbol{U}, \boldsymbol{V})_{\Omega^c} \\ &= \mathcal{B}_1(\boldsymbol{U}, \boldsymbol{V}) + \mathcal{I}(\boldsymbol{U}, \boldsymbol{V}). \end{aligned} \tag{4.7}$$

Here

$$\mathcal{B}_1(\boldsymbol{U}, \boldsymbol{V}) = -(\eta{\boldsymbol{\mu}_b}^{-1}\, \boldsymbol{\nabla}\times \boldsymbol{U}, \boldsymbol{\nabla}\times\boldsymbol{V})_{\Omega^c} + k^2(d_0^2\boldsymbol{U}, \boldsymbol{V})_{\Omega^c}$$

and

$$\mathcal{I}(\boldsymbol{U}, \boldsymbol{V}) = -({\boldsymbol{\mu}_b}^{-1}\, \boldsymbol{\nabla}\times \boldsymbol{U}, (\nabla\bar{\eta}) \times \boldsymbol{V})_{\Omega_2} - k^2((d_0^2\boldsymbol{I} - \eta\boldsymbol{\mu}_b)\boldsymbol{U}, \boldsymbol{V})_{\Omega_2}.$$

The last two integrations are over Ω_2 since both $\nabla\bar{\eta}$ and $(d_0^2\boldsymbol{I} - \eta\boldsymbol{\mu}_b)$ vanish for $\rho > r_2$. We obviously have

$$|\mathcal{I}(\boldsymbol{V}, \boldsymbol{V})| \le C\|\boldsymbol{V}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_2)}\|\boldsymbol{V}\|_{\boldsymbol{L}^2(\Omega_2)}. \tag{4.8}$$

Choosing α as in (3.11) we obtain for $\boldsymbol{V} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega^c)$

$$(1 + \alpha^2)^{1/2}|\mathcal{B}_1(\boldsymbol{V}, \boldsymbol{V})| \ge |Re[(1 + i\alpha)\mathcal{B}_1(\boldsymbol{V}, \boldsymbol{V})]| \ge C\|\boldsymbol{V}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega^c)}. \tag{4.9}$$

Multiplication by $\bar{\eta}^{-1}$ is a bounded operator on $\boldsymbol{H}(\mathbf{curl}; \Omega^c)$. Thus,

$$\begin{aligned} \|\boldsymbol{W}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)} &\le C\bigg(\frac{|\mathcal{B}(\boldsymbol{W}, \boldsymbol{W})|}{\|\boldsymbol{W}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)}} + \|\boldsymbol{W}\|_{\boldsymbol{L}^2(\Omega_2)}\bigg) \\ &\le C\bigg(\sup_{\boldsymbol{V}\in \boldsymbol{H}_0(\mathbf{curl};\Omega^c)} \frac{|\mathcal{A}(\boldsymbol{W}, \boldsymbol{V})|}{\|\boldsymbol{V}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)}} + \|\boldsymbol{W}\|_{\boldsymbol{L}^2(\Omega_2)}\bigg). \end{aligned}$$

Let $\tilde{H}^1_0(\Omega^c)$ denote the completion $C_0^\infty(\Omega^c)$ in the norm $\|\nabla u\|_{\boldsymbol{L}^2(\Omega^c)}$. For $\boldsymbol{V} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega^c)$, we decompose $\boldsymbol{V}$ as $\boldsymbol{V} = \boldsymbol{v} + \nabla\psi$ with $\boldsymbol{v} \in \boldsymbol{X}(\Omega^c)$ and $\psi \in \tilde{H}^1_0(\Omega^c)$. Indeed, ψ is the solution of

$$(\boldsymbol{\mu}_b\nabla\psi, \nabla\theta) = (\boldsymbol{\mu}_b\boldsymbol{V}, \nabla\theta) \quad \text{for all } \theta \in \tilde{H}^1_0(\Omega^c). \tag{4.10}$$

This problem is uniquely solvable since the real part of $\boldsymbol{\mu}_b$ is uniformly positive definite. Moreover,

$$\|\nabla\psi\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)} \le C\|\boldsymbol{V}\|_{\boldsymbol{L}^2(\Omega^c)}$$

so

$$\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)} \le C\|\boldsymbol{V}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)}.$$

Thus, for $\boldsymbol{W} \in \boldsymbol{X}(\Omega^c)$

$$\begin{aligned}\|\boldsymbol{W}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)} &\leq C\left(\sup_{\boldsymbol{V}\in \boldsymbol{H}_0(\mathbf{curl};\Omega^c)} \frac{|\mathcal{A}(\boldsymbol{W},\boldsymbol{v})|}{\|\boldsymbol{V}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)}} + \|\boldsymbol{W}\|_{\boldsymbol{L}^2(\Omega_2)}\right)\\ &\leq C\left(\sup_{\boldsymbol{v}\in \boldsymbol{X}(\Omega^c)} \frac{|\mathcal{A}(\boldsymbol{W},\boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)}} + \|\boldsymbol{W}\|_{\boldsymbol{L}^2(\Omega_2)}\right).\end{aligned}$$

To apply Theorem 3.2 (with $T : \boldsymbol{X}(\Omega^c) \to \boldsymbol{L}^2(\Omega_2)$, the imbedding into $\boldsymbol{L}^2(\Omega_2)$ of the restriction of elements of $\boldsymbol{X}(\Omega^c)$ to Ω_2) to prove (4.6) we need only check the uniqueness property.

We note that we have the uniqueness properties for $\mathcal{A}$ and its adjoint in $\boldsymbol{H}_0(\mathbf{curl};\Omega^c)$ (cf. [13]), specifically, if $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega^c)$ satisfies

$$\mathcal{A}(\boldsymbol{u},\boldsymbol{w}) = 0 \quad \text{for all } \boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega^c) \tag{4.11}$$

then $\boldsymbol{u} = \mathbf{0}$ and if $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega^c)$ satisfies

$$\mathcal{A}(\boldsymbol{w},\boldsymbol{u}) = 0 \quad \text{for all } \boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega^c) \tag{4.12}$$

then $\boldsymbol{u} = \mathbf{0}$.

We show the above uniqueness property on the restricted space $\boldsymbol{X}(\Omega^c)$. Suppose that $\boldsymbol{W}$ is in $\boldsymbol{X}(\Omega^c)$ and satisfies

$$\mathcal{A}(\boldsymbol{W},\boldsymbol{v}) = 0 \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X}(\Omega^c). \tag{4.13}$$

For any $\boldsymbol{V} \in \boldsymbol{H}_0(\mathbf{curl};\Omega^c)$, we decompose $\boldsymbol{V} = \boldsymbol{v} + \nabla\psi$ as above. Since $\boldsymbol{W} \in \boldsymbol{X}(\Omega^c)$, $\mathcal{A}(\boldsymbol{W},\boldsymbol{V}) = \mathcal{A}(\boldsymbol{W},\boldsymbol{v}) = 0$ by (4.13) and $\boldsymbol{W} = \mathbf{0}$ follows from (4.11). Finally, (4.6) follows from Theorem 3.2.

We can now complete the proof of the theorem. For $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\Omega^c)$, define $\phi \in \tilde{H}^1_0(\Omega^c)$ by

$$\mathcal{A}(\nabla\phi,\nabla\psi) = k^2(\boldsymbol{\mu}_b\nabla\phi,\nabla\psi) = (\bar{\boldsymbol{\Phi}},\nabla\psi), \quad \text{for all } \psi \in \tilde{H}^1_0(\Omega^c).$$

Next define $\boldsymbol{W} \in \boldsymbol{X}(\Omega^c)$ to be the solution of

$$\mathcal{A}(\boldsymbol{W},\boldsymbol{v}) = (\bar{\boldsymbol{\Phi}},\boldsymbol{v}) - k^2(\boldsymbol{\mu}_b\nabla\phi,\boldsymbol{v}), \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X}(\Omega^c). \tag{4.14}$$

As above, the uniqueness property for the adjoint on $\boldsymbol{H}_0(\mathbf{curl};\Omega^c)$ implies the corresponding property on the restricted space $\boldsymbol{X}(\Omega^c)$. Thus, the existence of $\boldsymbol{W}$ satisfying (4.14) follows from (4.6) and the generalized Lax-Milgram Lemma.

The solution of (4.4) is then given by $\boldsymbol{z} = \boldsymbol{W} + \nabla\phi$. Indeed, for any $\boldsymbol{\Theta} \in \boldsymbol{H}_0(\mathbf{curl};\Omega^c)$, we decompose $\boldsymbol{\Theta} = \boldsymbol{v} + \nabla\psi$ with $\boldsymbol{v} \in \boldsymbol{X}(\Omega^c)$. Then

$$\begin{aligned}\mathcal{A}(\boldsymbol{z},\boldsymbol{\Theta}) &= \mathcal{A}(\boldsymbol{W},\boldsymbol{v}) + \mathcal{A}(\boldsymbol{W},\nabla\psi) + \mathcal{A}(\nabla\phi,\boldsymbol{v}) + \mathcal{A}(\nabla\phi,\nabla\psi)\\ &= (\bar{\boldsymbol{\Phi}},\boldsymbol{v}) - k^2(\boldsymbol{\mu}_b\nabla\phi,\boldsymbol{v}) + k^2(\boldsymbol{\mu}_b\nabla\phi,\boldsymbol{v}) + (\bar{\boldsymbol{\Phi}},\nabla\psi) = (\bar{\boldsymbol{\Phi}},\boldsymbol{\Theta}).\end{aligned}$$

Evidently

$$\|\boldsymbol{z}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)} \leq C\|\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\Omega^c)},$$

which concludes the proof. □

Corollary 4.1. *Let $\widetilde{\boldsymbol{E}}$ be the solution of (2.4) and $\widehat{\boldsymbol{g}}$ be an H-curl extension of $\boldsymbol{g}$ with support in Ω_0. Then*

$$\|\widetilde{\boldsymbol{E}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^c)} \leq C\|\widehat{\boldsymbol{g}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_0)}.$$

We can now prove the uniqueness theorem for the problem on Ω_∞.

Theorem 4.2. *For R_t sufficiently large, the only solution $\boldsymbol{\Theta} \in \boldsymbol{H}_0(\mathbf{curl};\Omega_\infty)$ satisfying (4.2) is $\boldsymbol{\Theta} = \mathbf{0}$.*

Proof. Suppose that $\boldsymbol{\Theta}$ satisfies (4.2). For $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\Omega^c)$ with support in Ω_4 let $\hat{\boldsymbol{z}}$ satisfy (4.3). Both $\boldsymbol{n}\times\boldsymbol{\Theta}$ and $\boldsymbol{n}\times\hat{\boldsymbol{z}}$ vanish on $\partial\Omega$. Also $\boldsymbol{n}\times\boldsymbol{\Theta}$ vanishes on Γ_∞. In addition, the components of $\hat{\boldsymbol{z}}$ satisfy (3.15) with $\beta = k^2 d_0^2$ outside of Ω_2 so $\hat{\boldsymbol{z}}$ is in $\boldsymbol{H}^2$ near Γ_∞. Thus,

$$(\boldsymbol{\Theta},\boldsymbol{\Phi})_{\Omega_4} = \mathcal{A}(\boldsymbol{\Theta},\hat{\boldsymbol{z}}) = d_0^{-1} < \boldsymbol{n}\times\boldsymbol{\nabla}\times\boldsymbol{\Theta},\hat{\boldsymbol{z}} >_{\Gamma_\infty} . \tag{4.15}$$

Let $\widetilde{\boldsymbol{H}}^1(\Omega_\infty\setminus\bar\Omega_3)$ denote the set of functions in $\boldsymbol{H}^1(\Omega_\infty\setminus\bar\Omega_3)$ which vanish on Γ_3. Set

$$\|\boldsymbol{w}\|_{\boldsymbol{H}^{-1/2}(\Gamma_\infty)} = \sup_{\boldsymbol{\phi}\in\widetilde{\boldsymbol{H}}^1(\Omega_\infty\setminus\bar\Omega_3)} \frac{|< \boldsymbol{w},\boldsymbol{\phi} >_{\Gamma_\infty}|}{\|\boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Omega_\infty\setminus\bar\Omega_3)}}.$$

Let χ be a smooth cutoff function with support $\bar D_1$ in S_γ (of Proposition 3.2) which is one on Γ_∞. Since each component of $\hat{\boldsymbol{z}}$ satisfies (3.15) with $\beta = k^2\bar d_0^2$, applying Propositions 3.1 and 3.2 and Theorem 4.1 it follows that

$$\begin{aligned}
|(\boldsymbol{\Theta},\boldsymbol{\Phi})_{\Omega_4}| &\le C\frac{|< \boldsymbol{n}\times\boldsymbol{\nabla}\times\boldsymbol{\Theta},\chi\hat{\boldsymbol{z}} >_{\Gamma_\infty}|}{\|\chi\hat{\boldsymbol{z}}\|_{\boldsymbol{H}^1(S_\gamma)}}\|\hat{\boldsymbol{z}}\|_{\boldsymbol{H}^1(D_1)}\\
&\le C\frac{|< \boldsymbol{n}\times\boldsymbol{\nabla}\times\boldsymbol{\Theta},\chi\hat{\boldsymbol{z}} >_{\Gamma_\infty}|}{\|\chi\hat{\boldsymbol{z}}\|_{\boldsymbol{H}^1(S_\gamma)}}\|\hat{\boldsymbol{z}}\|_{\boldsymbol{L}^2(S_\gamma)}\\
&\le Ce^{-\sigma_0 kR_t}\|\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\Omega_4)}\|\boldsymbol{n}\times(\boldsymbol{\nabla}\times\boldsymbol{\Theta})\|_{\boldsymbol{H}^{-1/2}(\Gamma_\infty)}.
\end{aligned} \tag{4.16}$$

We next estimate the negative norm on the right hand side above. Let $\hat{\boldsymbol{h}}$ be in $\widetilde{\boldsymbol{H}}^1(\Omega_\infty\setminus\bar\Omega_3)$ and $\boldsymbol{\psi}\in\boldsymbol{H}_0(\mathbf{curl};\Omega_\infty\setminus\bar\Omega_3)$ satisfy

$$-(\boldsymbol{\nabla}\times\boldsymbol{\psi},\boldsymbol{\nabla}\times\boldsymbol{\theta})_{\Omega_\infty\setminus\bar\Omega_3} + k^2\bar d_0^2(\boldsymbol{\psi},\boldsymbol{\theta})_{\Omega_\infty\setminus\bar\Omega_3} = -(\boldsymbol{\nabla}\times\hat{\boldsymbol{h}},\boldsymbol{\nabla}\times\boldsymbol{\theta})_{\Omega_\infty\setminus\bar\Omega_3} + k^2\bar d_0^2(\hat{\boldsymbol{h}},\boldsymbol{\theta})_{\Omega_\infty\setminus\bar\Omega_3}$$

for all $\boldsymbol{\theta}\in\boldsymbol{H}_0(\mathbf{curl};\Omega_\infty\setminus\bar\Omega_3)$. This problem is well posed since d_0^2 has a nonzero imaginary part. We set $\boldsymbol{h} = \hat{\boldsymbol{h}} - \boldsymbol{\psi}$. It follows that

$$\|\boldsymbol{h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_\infty\setminus\bar\Omega_3)} \le C\|\hat{\boldsymbol{h}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_\infty\setminus\bar\Omega_3)}.$$

Note that both $\boldsymbol{\Theta}$ and $\boldsymbol{h}$ satisfy homogeneous equations in $\Omega_\infty\setminus\bar\Omega_3$,

$$\begin{aligned}
-\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\Theta} + k^2 d_0^2\boldsymbol{\Theta} &= \Delta\boldsymbol{\Theta} + k^2 d_0^2\boldsymbol{\Theta} = \mathbf{0},\\
-\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{h} + k^2\bar d_0^2\boldsymbol{h} &= \mathbf{0}.
\end{aligned} \tag{4.17}$$

It follows that $\boldsymbol{\nabla}\times\boldsymbol{\Theta}$ and $\boldsymbol{\nabla}\times\boldsymbol{h}$ are also in $\boldsymbol{H}(\mathbf{curl};\Omega_\infty\setminus\bar\Omega_3)$. Now, integrating by parts gives

$$\begin{aligned}
< \boldsymbol{n}\times\boldsymbol{\nabla}\times\boldsymbol{\Theta},\hat{\boldsymbol{h}} >_{\Gamma_\infty} &=< \boldsymbol{\nabla}\times\boldsymbol{\Theta},\hat{\boldsymbol{h}}\times\boldsymbol{n} >_{\Gamma_\infty}\\
&= -(\boldsymbol{\nabla}\times\boldsymbol{\Theta},\boldsymbol{\nabla}\times\boldsymbol{h})_{\Omega_\infty\setminus\bar\Omega_3} + (\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\Theta},\boldsymbol{h})_{\Omega_\infty\setminus\bar\Omega_3}\\
&=< \boldsymbol{\Theta},\boldsymbol{n}\times\boldsymbol{\nabla}\times\boldsymbol{h} >_{\Gamma_3} -(\boldsymbol{\Theta},\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{h})_{\Omega_\infty\setminus\bar\Omega_3}\\
&\quad + (\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\Theta},\boldsymbol{h})_{\Omega_\infty\setminus\bar\Omega_3}\\
&=< \boldsymbol{\Theta},\boldsymbol{n}\times\boldsymbol{\nabla}\times\boldsymbol{h} >_{\Gamma_3} .
\end{aligned} \tag{4.18}$$

The first integration by parts formula is justified as $\boldsymbol{h} = \hat{\boldsymbol{h}} + \boldsymbol{\psi}$ and the formula holds for both terms. The second integration by parts above is justified because $\boldsymbol{n} \times \boldsymbol{\Theta}$ vanishes on Γ_∞ and $\boldsymbol{\Theta}$ is smooth in a neighborhood of Γ_3 since it satisfies (3.15) with $\beta = k^2 d_0^2$ there. Finally, using (3.16) (with D_1 a domain containing Γ_3 and whose closure is in $\Omega_4 \setminus \bar{\Omega}_2$),

$$\begin{aligned} | < \boldsymbol{\Theta}, \boldsymbol{n} \times \boldsymbol{\nabla}\times \boldsymbol{h} >_{\Gamma_3} | &\le C \|\boldsymbol{\Theta}\|_{\boldsymbol{H}^1(D_1)} \| \boldsymbol{\nabla}\times \boldsymbol{h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_\infty \setminus \bar{\Omega}_3)} \\ &\le C \|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega_4)} \|\hat{\boldsymbol{h}}\|_{\boldsymbol{H}^1(\Omega_\infty \setminus \bar{\Omega}_3)}. \end{aligned}$$

Combining the above results shows that

$$\|\boldsymbol{n} \times \boldsymbol{\nabla}\times \boldsymbol{\Theta}\|_{\boldsymbol{H}^{-1/2}(\Gamma_\infty)} \le C \|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega_4)}$$

and hence, using (4.16),

$$\|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega_4)} \le C e^{-\sigma_0 k R_t} \|\boldsymbol{\Theta}\|_{\boldsymbol{L}^2(\Omega_4)}.$$

It follows that $\boldsymbol{\Theta}$ vanishes on Ω_4 for R_t sufficiently large. In this case, unique continuation implies that $\boldsymbol{\Theta}$ vanishes on all of Ω_∞. This completes the proof of the theorem. □

5. Existence and convergence of solutions of the truncated electromagnetic PML problem (2.8)

The existence and convergence of solutions to the PML problem depend on the uniqueness result of the previous section. Accordingly, we shall assume that the hypotheses of Theorem 4.2 are satisfied throughout this section.

We are now in position to prove the existence theorem.

Theorem 5.1. *Let $\boldsymbol{g}$ admit an $\boldsymbol{H}(\mathbf{curl};\Omega_0)$-extension $\widehat{\boldsymbol{g}}$ supported in Ω_0. Then for R_t sufficiently large, the truncated PML problem (2.8) has a unique solution $\widetilde{\boldsymbol{E_t}}$.*

Proof. The theorem will follow if we show the existence of a solution to (4.1). Now that we have proved uniqueness of (4.2), we follow the proof of Theorem 4.1 with Ω^c replaced by Ω_∞. In exactly the same way we arrive at the analogous inf-sup condition,

$$\|\boldsymbol{W}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_\infty)} \le C \left(\sup_{\boldsymbol{V} \in \boldsymbol{X}(\Omega_\infty)} \frac{|\mathcal{A}(\boldsymbol{W}, \boldsymbol{V})|}{\|\boldsymbol{V}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_\infty)}} \right), \quad \text{for all } v \in \boldsymbol{X}(\Omega_\infty). \tag{5.1}$$

Following the proof of Theorem 4.1, we define $\phi \in H^1_0(\Omega_\infty)$ by

$$\mathcal{A}(\nabla\phi, \nabla\psi) = \mathcal{A}(\boldsymbol{B}\widehat{\boldsymbol{g}}, \nabla\psi), \quad \text{for all } \nabla\psi \in H^1_0(\Omega_\infty).$$

Clearly, $\mathcal{A}(\boldsymbol{\theta}, \boldsymbol{u}) = 0$ for all $\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl};\Omega_\infty)$ is the same as $\mathcal{A}(\bar{\boldsymbol{u}}, \boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl};\Omega_\infty)$. As in the proof of Theorem 4.1, if $\boldsymbol{u} \in \boldsymbol{X}(\Omega_\infty)$ and $\mathcal{A}(\boldsymbol{\theta}, \boldsymbol{u}) = 0$ for all $\boldsymbol{u} \in \boldsymbol{X}(\Omega_\infty)$ then $\boldsymbol{u} = \boldsymbol{0}$. The generalized Lax-Millgram Theorem shows that there is a unique $\boldsymbol{W} \in \boldsymbol{X}(\Omega_\infty)$ satisfying

$$\mathcal{A}(\boldsymbol{W}, \boldsymbol{v}) = \mathcal{A}(\boldsymbol{B}\widehat{\boldsymbol{g}}, \boldsymbol{v}) - k^2(\boldsymbol{\mu}_b \nabla\phi, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{X}(\Omega_\infty).$$

Then $\boldsymbol{\Xi} = \boldsymbol{W} + \nabla\phi$ satisfies (4.1). Setting $\widetilde{\boldsymbol{E_t}} = \widehat{\boldsymbol{g}} - \boldsymbol{B}^{-1}\boldsymbol{\Xi}$ concludes the proof.

□

Finally we want to estimate the error created in replacing the solution of the scattering problem (2.1) by the solution of the truncated PML problem (2.8). To do this, we compare the solutions of (2.8) and (2.4), since the solutions of (2.4) and (2.1) coincide in

Ω_1. The proof follows the arguments in the proof of uniqueness. We have the following convergence theorem.

Theorem 5.2. *Let* $\widetilde{\boldsymbol{E}}$ *be the solution of (2.4) and* $\widetilde{\boldsymbol{E}_t}$ *be the solution of (2.8). For* R_t *sufficiently large,*

$$\|\widetilde{\boldsymbol{E}_t} - \widetilde{\boldsymbol{E}}\|_{\boldsymbol{L}^2(\Omega_4)} \leq Ce^{-2\sigma_0 kR_t}\|\widehat{\boldsymbol{g}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_0)}. \tag{5.2}$$

Proof. Let $\widehat{\boldsymbol{F}} = \boldsymbol{B}(\widetilde{\boldsymbol{E}_t} - \widetilde{\boldsymbol{E}})$. We follow the proof of Theorem 4.2 replacing $\boldsymbol{\Theta}$ by $\widehat{\boldsymbol{F}}$. The main difference is that $\boldsymbol{n}\times\widehat{\boldsymbol{F}}$ does not vanish on Γ_∞.

Let $\boldsymbol{\Phi}$ and $\hat{\boldsymbol{z}}$ be as in the proof of Theorem 4.2. Then, as in (4.15),

$$\begin{aligned}(\widehat{\boldsymbol{F}}, \boldsymbol{\Phi})_{\Omega_4} &= -(\widehat{\boldsymbol{F}}, \boldsymbol{\nabla}\times(\boldsymbol{\mu}_b^*)^{-1}\,\boldsymbol{\nabla}\times\hat{\boldsymbol{z}})_{\Omega_\infty} + k^2(\widehat{\boldsymbol{F}}, \boldsymbol{\mu}_b^*\hat{\boldsymbol{z}})_{\Omega_\infty}\\ &= -(\boldsymbol{\mu}_b{}^{-1}\,\boldsymbol{\nabla}\times\widehat{\boldsymbol{F}}, \boldsymbol{\nabla}\times\hat{\boldsymbol{z}})_{\Omega_\infty} + k^2(\boldsymbol{\mu}_b\widehat{\boldsymbol{F}}, \hat{\boldsymbol{z}})_{\Omega_\infty}\\ &\qquad - d_0^{-1} < \boldsymbol{n}\times\widetilde{\boldsymbol{E}}, \boldsymbol{\nabla}\times\hat{\boldsymbol{z}} >_{\Gamma_\infty}.\\ &= d_0^{-1} < \boldsymbol{n}\times\boldsymbol{\nabla}\times\widehat{\boldsymbol{F}}, \hat{\boldsymbol{z}} >_{\Gamma_\infty} - d_0^{-1} < \boldsymbol{n}\times\widetilde{\boldsymbol{E}}, \boldsymbol{\nabla}\times\hat{\boldsymbol{z}} >_{\Gamma_\infty}.\end{aligned} \tag{5.3}$$

Here $\boldsymbol{\mu}_b^*$ denotes the conjugate transpose.

To bound the first term on the right hand side of (5.3), we follow the proof of Theorem 4.2. The integration by parts on (4.18) gives an extra term, i.e.,

$$< \boldsymbol{n}\times\boldsymbol{\nabla}\times\widehat{\boldsymbol{F}}, \hat{\boldsymbol{h}} >_{\Gamma_\infty} = < \widehat{\boldsymbol{F}}, \boldsymbol{n}\times\boldsymbol{\nabla}\times\boldsymbol{h} >_{\Gamma_3} - < \widetilde{\boldsymbol{E}}, \boldsymbol{n}\times\boldsymbol{\nabla}\times\boldsymbol{h} >_{\Gamma_\infty}. \tag{5.4}$$

For the second term of (5.4), we note that since $\boldsymbol{h}$ satisfies the homogeneous equation (4.17),

$$\|\boldsymbol{n}\times\boldsymbol{\nabla}\times\boldsymbol{h}\|_{\boldsymbol{H}^{-1/2}(\Gamma_\infty)} \leq C\|\boldsymbol{h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_\infty\setminus\bar{\Omega}_3)}$$

so by Proposition 3.1 and Proposition 3.2,

$$| < \widetilde{\boldsymbol{E}}, \boldsymbol{n}\times\boldsymbol{\nabla}\times\boldsymbol{h} >_{\Gamma_\infty} | \leq Ce^{-\sigma_0 kR_t}\|\widetilde{\boldsymbol{E}}\|_{\boldsymbol{L}^2(\Omega^c)}\|\boldsymbol{h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_\infty\setminus\bar{\Omega}_3)}. \tag{5.5}$$

Using (5.4), (5.5), Proposition 3.1 and Proposition 3.2 and following the proof of Theorem 4.2 (below (4.18)) gives

$$| < \boldsymbol{n}\times\boldsymbol{\nabla}\times\widehat{\boldsymbol{F}}, \hat{\boldsymbol{z}} >_{\Gamma_\infty} | \leq C\|\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\Omega_4)}(e^{-\sigma_0 kR_t}\|\widehat{\boldsymbol{F}}\|_{\boldsymbol{L}^2(\Omega_4)} + e^{-2\sigma_0 kR_t}\|\widetilde{\boldsymbol{E}}\|_{\boldsymbol{L}^2(\Omega_4)}). \tag{5.6}$$

Finally we bound the second term on the right hand side of (5.3). Using a trace inequality we have that

$$| < \boldsymbol{n}\times\widetilde{\boldsymbol{E}}, \boldsymbol{\nabla}\times\hat{\boldsymbol{z}} >_{\Gamma_\infty} | \leq C\|\widetilde{\boldsymbol{E}}\|_{\boldsymbol{H}^1(S_\gamma)}\|\hat{\boldsymbol{z}}\|_{\boldsymbol{H}^2(S_\gamma)}.$$

Note that the components $\widetilde{\boldsymbol{E}}$ and $\hat{\boldsymbol{z}}$ satisfy (3.15) with $\beta = k^2 d_0^2$ in S_γ. Thus from Proposition 3.1, Theorem 4.1 and Proposition 3.2, we obtain

$$| < \boldsymbol{n}\times\widetilde{\boldsymbol{E}}, \boldsymbol{\nabla}\times\hat{\boldsymbol{z}} >_{\Gamma_\infty} | \leq Ce^{-2\sigma_0 kR_t}\|\widetilde{\boldsymbol{E}}\|_{\boldsymbol{L}^2(\Omega^c)}\|\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\Omega_4)}.$$

Combining the above gives

$$|(\widehat{\boldsymbol{F}}, \boldsymbol{\Phi})_{\Omega_4}| \leq C\|\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\Omega_4)}(e^{-2\sigma_0 kR_t}\|\widetilde{\boldsymbol{E}}\|_{\boldsymbol{L}^2(\Omega^c)} + e^{-\sigma_0 kR_t}\|\widehat{\boldsymbol{F}}\|_{\boldsymbol{L}^2(\Omega_4)}).$$

The theorem easily follows from the above inequality and Corollary 4.1. □

6. Appendix

We now provide a proof of Lemma 4.1. We first consider the case of Ω^c and $\omega = \Omega_2$. Let χ be a smooth cutoff function which is one on $\Omega_2 \setminus \Omega_1$ and supported in $\Omega_3 \setminus \Omega_0$. Let $\boldsymbol{W}$ be in $\boldsymbol{X}(\Omega^c)$ and set $\boldsymbol{W}_1 = (1-\chi)\boldsymbol{W}$. Then, $\boldsymbol{W}_1$ is in $\boldsymbol{H}_0(\mathbf{curl};\Omega_2) \cap \boldsymbol{H}(\mathrm{div};\Omega_2)$ and therefore is in $\boldsymbol{H}^s(\Omega_2)$ (see [1]).

The proof in this case will be complete if we show that $\boldsymbol{W}$ is in $\boldsymbol{H}^s(D)$ where $D = \Omega_3 \setminus \Omega_0$. Let $\check{D}$ and $\tilde{D}$ extend D ($D \subset \tilde{D} \subset \check{D}$) with the closure of $\check{D}$ contained in Ω^c and let χ_1 be a cutoff function which is supported on $\check{D}$ and is one on $\tilde{D}$. Let $\phi \in H_0^1(\check{D})$ be the solution of

$$(\nabla\phi, \nabla\theta)_{\check{D}} = (\chi_1 \boldsymbol{W}, \nabla\theta)_{\check{D}} \quad \text{for all } \theta \in H_0^1(\check{D}).$$

Then $\widetilde{\boldsymbol{W}} = \chi_1 \boldsymbol{W} - \nabla\phi$ is in $\boldsymbol{H}_0(\mathbf{curl};\check{D}) \cap \boldsymbol{H}(\mathrm{div};\check{D})$, i.e., it is in $\boldsymbol{H}^s(\check{D})$. We note that ϕ also satisfies

$$(\boldsymbol{\mu}_b \nabla\phi, \nabla\theta)_{\tilde{D}} = -(\boldsymbol{\mu}_b \widetilde{\boldsymbol{W}}, \nabla\theta)_{\tilde{D}} \quad \text{for all } \theta \in H_0^1(\tilde{D}). \tag{6.1}$$

Now, $\widetilde{\boldsymbol{W}}$ is in $\boldsymbol{H}^1(\tilde{D})$ (see, Corollary 2.10 of [7]) so the right hand side above coincides with a bounded functional on L^2. Since the coefficients in $\boldsymbol{\mu}_b$ are in $W^2_\infty(\tilde{D})$, the solution ϕ is in $H^2(D)$, i.e., $\nabla\phi$ is in $\boldsymbol{H}^1(D)$. Thus, $\boldsymbol{W} = \widetilde{\boldsymbol{W}} + \nabla\phi$ is in $\boldsymbol{H}^1(D)$.

The proof in the case of Ω_∞ is similar. The only difference is that one uses the constant coefficient operator in the neighborhood of both the inner and outer boundary (of Ω_∞) to reduce to regularity on an overlapping interior domain.

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