COMPUTATIONAL SCALES OF SOBOLEV NORMS WITH APPLICATION TO PRECONDITIONING

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ABSTRACT. This paper provides a framework for developing computationally efficient multilevel preconditioners and representations for Sobolev norms. Specifically, given a Hilbert space V and a nested sequence of subspaces, $V_1 \subset V_2 \subset \ldots \subset V$, we construct operators which are spectrally equivalent to those of the form $\mathcal{A} = \sum_k \mu_k (Q_k - Q_{k-1})$. Here μ_k, $k = 1, 2, \ldots$ are positive numbers and Q_k is the orthogonal projector onto V_k with $Q_0 = 0$. We first present abstract results which show when $\mathcal{A}$ is spectrally equivalent to a similarly constructed operator $\widetilde{\mathcal{A}}$ defined in terms of an approximation $\widetilde{Q}_k$ of Q_k , for $k = 1, 2, \ldots$.

We show that these results lead to efficient preconditioners for discretizations of differential and pseudo-differential operators of positive and negative order. These results extend to sums of operators. For example, singularly perturbed problems such as $I - \epsilon\Delta$ can be preconditioned uniformly independently of the parameter ϵ. We also show how to precondition an operator which results from Tikhonov regularization of a problem with noisy data. Finally, we describe how the technique provides computationally efficient bounded discrete extensions which have applications to domain decomposition.

1. INTRODUCTION

Multilevel subspace decompositions provide tools for the construction of preconditioners. One of the first examples of such a construction was provided in [3] where a simple additive multilevel operator (BPX) was developed for preconditioning second order elliptic boundary value problems. This preconditioner was defined in terms of a nested sequence of multilevel piecewise linear and continuous approximation spaces $V_1 \subset V_2 \subset \ldots \subset V_J$. The analysis of the BPX preconditioner involves the verification of norm equivalences of the form

$$\|u\|^2_{H^1(\Omega)} \simeq \sum_{k=1}^{J} h_k^{-2} \|(Q_k - Q_{k-1})u\|^2_{L^2(\Omega)}, \qquad \text{for all } u \in V_J. \tag{1.1}$$

The above norms are those corresponding to the Sobolev space $H^1(\Omega)$ and $L^2(\Omega)$ respectively, Q_k denotes the $L^2(\Omega)$ orthogonal projection onto V_k and $Q_0 = 0$. The quantity h_k is the approximation parameter associated with V_k. The original results in [3] were sharpened by [13] and [20] to show that (1.1) holds with constants of equivalence independent

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of J. Practical preconditioners involve the replacement of the operator $Q_k - Q_{k-1}$ by easily computable operators as discussed in [3].

In addition to the above application, there are other practical applications of multilevel decompositions. In particular, for boundary element methods, it is important to have computationally simple operators which are equivalent to pseudo-differential operators of order one and minus one. In addition, multilevel decompositions which provide norm equivalences for $H^{1/2}(\partial\Omega)$ can be used to construct bounded extension operators used in nonoverlapping domain decomposition with inexact subdomain solves.

The equivalence (1.1) is the starting point of the multilevel analysis. This inequality is valid for $J = \infty$ in which case we get a norm equivalence on $H^1(\Omega)$. It follows from (1.1) that

$$\|v\|^2_{H^s(\Omega)} \simeq \sum_{k=1}^{\infty} h_k^{-2s} \|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)},$$

for $s \in [0,1]$. Here $\|\cdot\|_{H^s(\Omega)}$ denotes the norm on the Sobolev space $H^s(\Omega)$ of order s. This means that the operator

$$\mathcal{A}^s = \sum_{k=1}^{\infty} h_k^{-2s} (Q_k - Q_{k-1}) \tag{1.2}$$

can be used in preconditioning applications. However, $\mathcal{A}^s$ is somewhat expensive to evaluate since the evaluation of the projector Q_k requires the solution of a Gram matrix problem. Thus, many researchers have sought computationally efficient operators which are equivalent to $\mathcal{A}^s$.

Some techniques for constructing such operators based on wavelet or wavelet–like space decompositions are given by [5], [9], [10], [15], [16], [18], [19] and others. In the domain decomposition literature, extension operators that exploit multilevel decomposition were used in [4], [8], and [12].

In this paper, we construct simple multilevel decomposition preconditioning operators which can also be used to define norms equivalent to the usual norms on Sobolev spaces. Specifically, we develop computationally efficient operators which are uniformly equivalent to the more general operator

$$\mathcal{A}_J = \sum_{k=1}^{J} \mu_k (Q_k - Q_{k-1}), \tag{1.3}$$

where $1 \le J \le \infty$ and $\{\mu_k\}$ are positive constants. We start by proving an abstract theorem. Let $\{\widetilde{Q}_k\}$, with $\widetilde{Q}_k : V_J \to V_k$, be another sequence of linear operators. The theorem shows that the operators $\mathcal{A}_J$ and

$$\widetilde{\mathcal{A}}_J = \sum_{k=1}^{J} \mu_k (\widetilde{Q}_k^t - \widetilde{Q}_{k-1}^t)(\widetilde{Q}_k - \widetilde{Q}_{k-1}) \tag{1.4}$$

are spectrally equivalent under appropriate assumptions on the spaces V_k, the operators $\widetilde{Q}_k$ and the sequence $\{\mu_k\}$. Here $\widetilde{Q}_k^t$ is the adjoint of $\widetilde{Q}_k$. The abstract results are subsequently applied to develop efficient preconditioners when $\widetilde{Q}_k$ is defined in terms of a simple averaging operator. Some partial results involving the operator used here were stated by Nepomnyaschikh [12].

Because of the generality of the abstract results, they can be applied to preconditioning sums of operators. An example of this is the so-called "singularly perturbed" problem resulting from preconditioning parabolic time stepping problems which leads to

$$\mu_k = (\epsilon h_k^{-2} + 1)^{-1}.$$

Here ϵ is the time step size. Our results give rise to preconditioned systems with uniformly bounded condition numbers independent of the parameter ϵ.

We note that [16] provides an L^2-stable local basis for the spaces $\{\text{Range}(Q_k - Q_{k-1})\}$. With such a construction it is possible to obtain preconditioners for the applications considered in this paper. However, our approach is somewhat simpler to implement. In addition, our abstract framework allows for easy application to other situations such as function spaces which are piecewise quadratic.

An outline of the remainder of the paper is as follows. Section 2 gives an abstract framework for norm approximation in a Hilbert space setting along with an abstract theorem which provides equivalence estimates comparing (1.3) and (1.4). In Section 3 we give an example of a sequence of computationally efficient operators, $\{\widetilde{Q}_k\}$, in the case of polygonal domains, and verify that they satisfy the hypotheses required for application of the abstract theory. In Section 4 we discuss some applications. Finally, the results of numerical experiments which illustrate the effectiveness of the preconditioners are reported in Section 5.

2. A NORM EQUIVALENCE THEOREM

In this section, we provide abstract conditions which imply the spectral equivalence of (1.3) and (1.4). We start by introducing the multilevel spaces. Let V be a Hilbert space with inner product $(\cdot, \cdot)$. We assume that we are given a nested sequence of approximation subspaces,

$$V_1 \subset V_2 \subset \ldots \subset V,$$

and that this sequence is dense in V. Let θ_j, $j = 1, 2, \ldots$, be a non-decreasing sequence of positive real numbers. Define H to be the subspace of V such that the norm

$$|||v||| = \left(\sum_{j=1}^{\infty} \theta_j \|(Q_j - Q_{j-1})v\|^2 \right)^{1/2}$$

is finite. Here $\|\cdot\|$ denotes the norm in V, Q_j for $j > 0$, denotes the orthogonal projection onto V_j and $Q_0 = 0$. Clearly H is a Hilbert space and $\{V_k\}$ is dense in H.

The following properties are obvious from the construction.

1. The "inverse inequality" holds for V_j, i.e.,

$$|||v||| \le \theta_j^{1/2} \|v\|, \qquad \text{for all } v \in V_j. \tag{2.1}$$

2. The "approximation property" holds for V_j, i.e.,

$$\|(Q_j - Q_{j-1})v\| \le \theta_j^{-1/2} |||v|||, \qquad \text{for all } v \in H. \tag{2.2}$$

As discussed in the introduction, the abstract results will be stated in terms of an additional sequence of "approximation" operators, $\widetilde{Q}_k : V \to V_k$ for $k > 0$ and $\widetilde{Q}_0 = 0$. These operators are assumed to satisfy the following three conditions, for $k = 1, 2, \ldots$.

1. An "approximation property": There exists a constant C_A such that

$$\|(Q_k - \widetilde{Q}_k)v\| \le C_A \theta_k^{-1/2} |||v|||, \quad \text{for all } v \in H. \tag{2.3}$$

2. Uniform coercivity of $\widetilde{Q}_k$: There exists a $\delta > 0$ such that

$$\delta \|v_k\|^2 \le (\widetilde{Q}_k v_k, v_k), \quad \text{for all } v_k \in V_k. \tag{2.4}$$

3. The range of $\widetilde{Q}_k^t$, the adjoint of $\widetilde{Q}_k$, is contained in V_k. This condition is equivalent to

$$\widetilde{Q}_k Q_k = Q_k \widetilde{Q}_k. \tag{2.5}$$

Remark 2.1. *Let $\{\phi_i\}_{i=1}^m$ be a basis for V_k. It is not difficult to see that there exists $\{f_i\}_{i=1}^m$ with $f_i \in V$ such that*

$$\tilde{Q}_k v = \sum_{i=1}^m (v, f_i)\ \phi_i \quad \text{for all } v \in V.$$

Then

$$\tilde{Q}_k^t w = \sum_{i=1}^m (w, \phi_i)\ f_i \quad \text{for all } w \in V_k.$$

Thus Condition 3 above holds if and only if $f_i \in V_k$, for $i = 1, \ldots, m$.

The purpose of this section is to provide abstract conditions which guarantee that the symmetric operators $\mathcal{A}_J$ and $\widetilde{\mathcal{A}}_J$, defined respectively by (1.3) and (1.4), are spectrally equivalent. Let $\mathcal{L} = (\ell_{k,j})$ be the lower triangular (infinite) matrix with nonzero entries

$$\ell_{k,j} = \left(\frac{\theta_j \mu_k}{\theta_k \mu_j} \right)^{1/2}, \ k \ge j. \tag{2.6}$$

We assume that $\mathcal{L}$ has bounded l_2 norm, i.e.,

$$\|\mathcal{L}\|_{\ell_2} \equiv \sup_{\{\xi_k\}, \{\zeta_k\}} \frac{\sum\limits_{k=1}^{\infty} \sum\limits_{j \le k} \ell_{k,j}\ \xi_k \zeta_j}{\left(\sum\limits_{k=1}^{\infty} \xi_k^2 \right)^{1/2} \left(\sum\limits_{k=1}^{\infty} \zeta_k^2 \right)^{1/2}} \le C_{\mathcal{L}}. \tag{2.7}$$

The above condition implies that

$$\mu_k \le C \theta_k$$

for $C = C_{\mathcal{L}}^2 \mu_1 / \theta_1$. Thus, $(\mathcal{A}_J v, v) < \infty$ for all $v \in H$.

We introduce one final condition: There exists a constant α such that

$$\mu_k + \mu_{k+1} \le \alpha \mu_k, \ \text{for } k = 1, 2, \ldots . \tag{2.8}$$

We can now state the main abstract theorem.

Theorem 2.1. *Assume that conditions (2.3)–(2.5), (2.7), and (2.8) are satisfied. Then the operator $\widetilde{\mathcal{A}}_J$ defined by (1.4), with $1 \le J \le \infty$, satisfies*

$$\begin{aligned} [3(1 + \alpha \delta^{-2} C_A^2 C_{\mathcal{L}}^2)]^{-1} (\mathcal{A}_J v, v) &\le (\widetilde{\mathcal{A}}_J v, v) \\ &\le 3(1 + \alpha C_A^2 C_{\mathcal{L}}^2)\ (\mathcal{A}_J v, v), \quad \text{for all } v \in H. \end{aligned}$$

Remark 2.2. *If W is the completion of H under the norm $\|v\|_{\mathcal{A}} = (\mathcal{A}_\infty v, v)^{1/2}$, then the estimate of Theorem 2.1 extends to all of W by density.*

For the purpose of proving the theorem, we now prove the following lemma.

Lemma 2.1. *Assume that conditions (2.3)–(2.5), and (2.7) are satisfied. Then for all $u \in H$,*

$$\sum_{k=1}^{J} \mu_k \|(Q_k - \widetilde{Q}_k)u\|^2 \leq C_A^2 C_{\mathcal{L}}^2 \, (\mathcal{A}_J u, u) \tag{2.9}$$

and

$$\sum_{k=1}^{J} \mu_k \|(Q_k - \widetilde{Q}_k)u\|^2 \leq \delta^{-2} C_A^2 C_{\mathcal{L}}^2 \, (\widetilde{\mathcal{A}}_J u, u). \tag{2.10}$$

Proof. By (2.5), for all $u \in H$,

$$\begin{aligned} \sum_{k=1}^{J} \mu_k \|(Q_k - \widetilde{Q}_k)u\|^2 &= \sum_{k=1}^{J} \mu_k \left((Q_k - \widetilde{Q}_k)Q_k u, (Q_k - \widetilde{Q}_k)u\right) \\ &= \sum_{k=1}^{J} \sum_{j=1}^{k} \mu_k \left((Q_k - \widetilde{Q}_k)(Q_j - Q_{j-1})u, (Q_k - \widetilde{Q}_k)u\right). \end{aligned} \tag{2.11}$$

In addition, by (2.4) and (2.5),

$$\begin{aligned} \sum_{k=1}^{J} \mu_k \|(Q_k - \widetilde{Q}_k)u\|^2 &\leq \delta^{-1} \sum_{k=1}^{J} \mu_k \left(\widetilde{Q}_k (Q_k - \widetilde{Q}_k)u, (Q_k - \widetilde{Q}_k)u\right) \\ &= \delta^{-1} \sum_{k=1}^{J} \mu_k \left((Q_k - \widetilde{Q}_k)\widetilde{Q}_k u, (Q_k - \widetilde{Q}_k)u\right) \\ &= \delta^{-1} \sum_{k=1}^{J} \sum_{j=1}^{k} \mu_k \left((Q_k - \widetilde{Q}_k)(\widetilde{Q}_j - \widetilde{Q}_{j-1})u, (Q_k - \widetilde{Q}_k)u\right). \end{aligned} \tag{2.12}$$

The quantities on the right hand side of (2.11) and of (2.12) can be written as

$$\sum_{k=1}^{J} \sum_{j=1}^{k} \mu_k \left((Q_k - \widetilde{Q}_k)v_j, (Q_k - \widetilde{Q}_k)u\right) \tag{2.13}$$

by setting v_j equal to $(Q_j - Q_{j-1})u$ and $(\widetilde{Q}_j - \widetilde{Q}_{j-1})u$, respectively. Using (2.1) and (2.3), the quantity (2.13) is bounded by

$$\begin{aligned}
&\sum_{k=1}^{J}\sum_{j=1}^{k} \mu_k\left((Q_k - \widetilde{Q}_k)v_j, (Q_k - \widetilde{Q}_k)u\right) \\
&\qquad\qquad \le C_A \sum_{k=1}^{J}\sum_{j=1}^{k} \mu_k \theta_k^{-1/2} |||v_j|||\, \|(Q_k - \widetilde{Q}_k)u\| \\
&\qquad\qquad \le C_A \sum_{k=1}^{J}\sum_{j=1}^{k} \mu_k (\theta_j/\theta_k)^{1/2} \|v_j\| \|(Q_k - \widetilde{Q}_k)u\| \\
&\qquad\qquad = C_A \sum_{k=1}^{J}\sum_{j=1}^{k} \ell_{k,j}\, (\mu_j^{1/2}\|v_j\|)(\mu_k^{1/2}\|(Q_k - \widetilde{Q}_k)u\|).
\end{aligned} \tag{2.14}$$

It immediately follows from (2.7) that

$$\begin{aligned}
&\sum_{k=1}^{J}\sum_{j=1}^{k} \mu_k\left((Q_k - \widetilde{Q}_k)v_j, (Q_k - \widetilde{Q}_k)u\right) \\
&\qquad\qquad \le C_A C_{\mathcal{L}} \Big(\sum_{k=1}^{J} \mu_k \|v_k\|^2\Big)^{1/2} \Big(\sum_{k=1}^{J} \mu_k \|(Q_k - \widetilde{Q}_k)u\|^2\Big)^{1/2}.
\end{aligned} \tag{2.15}$$

Combining (2.11) and (2.15) gives

$$\sum_{k=1}^{J} \mu_k \|(Q_k - \widetilde{Q}_k)u\|^2 \le C_A C_{\mathcal{L}} \Big(\sum_{k=1}^{J} \mu_k \|(Q_k - Q_{k-1})u\|^2\Big)^{1/2} \Big(\sum_{k=1}^{J} \mu_k \|(Q_k - \widetilde{Q}_k)u\|^2\Big)^{1/2}.$$

The inequality (2.9) follows by obvious manipulations and (2.10) follows in a similar manner. □

Proof of Theorem 2.1. Note that $(\widetilde{Q}_k - \widetilde{Q}_{k-1}) = (Q_k - Q_{k-1}) - (Q_k - \widetilde{Q}_k) + (Q_{k-1} - \widetilde{Q}_{k-1})$. Thus for $v \in H$,

$$\begin{aligned}
(\widetilde{\mathcal{A}}_J v, v) &= \sum_{k=1}^{J} \mu_k \|(\widetilde{Q}_k - \widetilde{Q}_{k-1})v\|^2 \\
&\le 3\Big(\sum_{k=1}^{J} \mu_k \|(Q_k - Q_{k-1})v\|^2 + \sum_{k=1}^{J} (\mu_k + \mu_{k+1}) \|(Q_k - \widetilde{Q}_k)v\|^2\Big) \\
&\le 3(1 + \alpha C_A^2 C_{\mathcal{L}}^2)\,(\mathcal{A}_J v, v).
\end{aligned}$$

We used (2.8) and Lemma 2.1 for the last inequality above. The proof for the other inequality is essentially the same. This completes the proof of the theorem. □

2.1. **Development of preconditioners.** The above results can be applied to the development of preconditioners. Indeed, consider preconditioning an operator on V_J which is spectrally equivalent to

$$L_J = \sum_{k=1}^{J} \mu_k^{-1}(Q_k - Q_{k-1}). \tag{2.16}$$

Our preconditioner B_J is to be spectrally equivalent to the operator

$$A_J \equiv L_J^{-1} = \sum_{k=1}^{J} \mu_k(Q_k - Q_{k-1}).$$

Let

$$B_J = \sum_{k=1}^{J} \mu_k(\widetilde{Q}_k - \widetilde{Q}_{k-1})^t(\widetilde{Q}_k - \widetilde{Q}_{k-1}). \tag{2.17}$$

Then B_J and A_J are spectrally equivalent provided that $\{\mu_k\}$ and $\{\widetilde{Q}_k\}$ satisfy the hypothesis of the theorem. It follows that $B_J L_J$ is well conditioned.

2.2. **Preconditioning sums of operators.** We next consider the case of preconditioning sums of operators. Suppose $\{\hat{\mu}_k\}$ is another sequence which satisfies conditions (2.7) and (2.8). Then

$$\hat{L}_J = \sum_{k=1}^{J} \hat{\mu}_k^{-1}(Q_k - Q_{k-1}). \tag{2.18}$$

can be preconditioned by the operator defined by replacing μ_k by $\hat{\mu}_k$ in (2.17) above. The following corollary shows that the result can be extended to non-negative combinations of L_J and $\hat{L}_J$.

Corollary 2.1. *Assume that conditions (2.3)–(2.5) are satisfied and that (2.7) and (2.8) hold for both $\{\mu_k\}$ and $\{\hat{\mu}_k\}$. For nonnegative c_1, c_2 with $c_1 + c_2 > 0$ define*

$$B_J = \sum_{k=1}^{J} (c_1\mu_k^{-1} + c_2\hat{\mu}_k^{-1})^{-1}(\widetilde{Q}_k - \widetilde{Q}_{k-1})^t(\widetilde{Q}_k - \widetilde{Q}_{k-1}). \tag{2.19}$$

Then for $1 \le J \le \infty$,

$$\begin{aligned}[3(1 + 4\alpha\delta^{-2}C_A^2C_{\mathcal{L}}^2)]^{-1}((c_1L_J + c_2\hat{L}_J)^{-1}v, v) &\le (B_Jv, v)\\ &\le 3(1 + 4\alpha C_A^2C_{\mathcal{L}}^2)\,((c_1L_J + c_2\hat{L}_J)^{-1}v, v), \quad \text{for all } v \in H.\end{aligned}$$

The above corollary shows that B_J is spectrally equivalent to $(c_1L_J + c_2\hat{L}_J)^{-1}$ and hence provides a uniform preconditioner for $c_1L_J + c_2\hat{L}_J$. Moreover, the resulting condition number (for the preconditioned system) is bounded independently of the parameters c_1 and c_2.

Proof. Note that

$$(c_1L_J + c_2\hat{L}_J)^{-1} = \sum_{k=1}^{J} (c_1\mu_k^{-1} + c_2\hat{\mu}_k^{-1})^{-1}(Q_k - Q_{k-1}).$$

To apply the theorem to this operator, we simply must check the conditions on the sequence $\widetilde{\mu}_k = (c_1\mu_k^{-1} + c_2\hat{\mu}_k^{-1})^{-1}$. The corresponding lower triangular matrix has entries

$$\begin{aligned}(\widetilde{\mathcal{L}})_{k,j} = \left(\frac{\theta_j\widetilde{\mu}_k}{\theta_k\widetilde{\mu}_j}\right)^{1/2} &= \left(\frac{\theta_j(c_1\mu_j^{-1} + c_2\hat{\mu}_j^{-1})}{\theta_k(c_1\mu_k^{-1} + c_2\hat{\mu}_k^{-1})}\right)^{1/2} \\ &\le \left(\frac{\theta_j}{\theta_k}\left(\frac{\mu_k}{\mu_j} + \frac{\hat{\mu}_k}{\hat{\mu}_j}\right)\right)^{1/2} \le \left(\frac{\theta_j\mu_k}{\theta_k\mu_j}\right)^{1/2} + \left(\frac{\theta_j\hat{\mu}_k}{\theta_k\hat{\mu}_j}\right)^{1/2} = (\mathcal{L} + \hat{\mathcal{L}})_{k,j}.\end{aligned}$$

Since $0 \le (\widetilde{\mathcal{L}})_{k,j} \le (\mathcal{L} + \hat{\mathcal{L}})_{k,j}$, for every pair k, j, it follows that

$$\|\widetilde{\mathcal{L}}\|_{\ell_2} \le \|\mathcal{L} + \hat{\mathcal{L}}\|_{\ell_2} \le 2C_{\mathcal{L}}.$$

Because (2.8) holds for both $\{\mu_k\}$ and $\{\hat{\mu}_k\}$, it clearly holds for $\{\widetilde{\mu}_k\}$. The corollary follows by application of the theorem. □

3. A SIMPLE APPROXIMATION OPERATOR $\widetilde{Q}_k$

In this section, we define and analyze a simple approximation operator $\widetilde{Q}_k$. Our applications involve Sobolev spaces with possibly mixed boundary conditions.

Let Ω be a polygonal domain in R^2 with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ where Γ_D and Γ_N are essentially disjoint. Dirichlet boundary conditions are imposed on Γ_D. We consider domains in R^2 for convenience. Generalizations of the results to be presented to domains in R^d, with $d > 2$, at least for rectangular parallelepipeds, are straightforward.

For non-negative integers s, let $H^s(\Omega)$ denote the Sobolev space of order s on Ω (see, e.g. [6],[7]). The corresponding norm and semi-norm are denoted $\|\cdot\|_{H^s(\Omega)}$ and $|\cdot|_{H^s(\Omega)}$ respectively. The space $H^1_D(\Omega)$ is defined to be the functions in $H^1(\Omega)$ which vanish on Γ_D and for $s > 1$, $H^s_D(\Omega) = H^s(\Omega) \cap H^1_D(\Omega)$. For positive non-integers s, the spaces $H^s(\Omega)$ and $H^s_D(\Omega)$ are defined by interpolation between the neighboring integers using the real method of Lions and Peetre (cf. [7]). For negative s, $H^s(\Omega)$ is defined to be the space of linear functionals for which the norm

$$\|u\|_{H^s(\Omega)} = \sup_{\phi \in H_D^{-s}(\Omega)} \frac{< u, \phi >}{\|\phi\|_{H_D^{-s}(\Omega)}}$$

is finite. Here $< \cdot, \cdot >$ denotes the duality pairing. Clearly, for $s < 0$, $L^2(\Omega) \subseteq H^s(\Omega)$ if we identify $u \in L^2(\Omega)$ with the functional $< u, \phi > \equiv (u, \phi)$.

3.1. **Some basic approximation properties.** Let $\mathcal{T}$ be a locally quasi-uniform triangulation of Ω and τ be a closed triangle in $\mathcal{T}$ with diameter h_τ. Let $\widetilde{\tau}$ be the subset of the triangles in $\mathcal{T}$ whose boundaries intersect τ and define $V_{\widetilde{\tau}}$ to be the finite element approximation subspace consisting of functions which are continuous on $\widetilde{\tau}$ and piecewise linear with respect to the triangles of $\widetilde{\tau}$. Note that there are no boundary conditions imposed on the elements of $V_{\widetilde{\tau}}$. We restrict the discussion in this paper to piecewise linear subspaces. Extensions to more general nodal finite element subspaces pose no significant additional difficulties.

The following facts are well known.

1. Given $u \in H^1(\widetilde{\tau})$, there exists a constant $\widetilde{u}$ such that

$$\|u - \widetilde{u}\|_{H^s(\widetilde{\tau})} \le Ch_\tau^{1-s}|u|_{H^1(\widetilde{\tau})}, \quad s = 0, 1. \tag{3.1}$$

2. Given $u \in H^2(\widetilde{\tau})$, there exists a linear function $\widetilde{u}$ such that

$$\|u - \widetilde{u}\|_{H^s(\widetilde{\tau})} \le C h_\tau^{2-s} |u|_{H^2(\widetilde{\tau})}, \quad s = 0, 1, 2. \tag{3.2}$$

The best constants satisfying the above inequalities clearly depend on the shape of the domain $\widetilde{\tau}$. However, under the assumption that the triangulation is locally quasi-uniform, it is possible to show that the above inequalities hold with constants only depending on s and on the quasi-uniformity constants.

For the purpose of analyzing our multilevel example we define the following local approximation operator $\widetilde{Q}_{\widetilde{\tau}} : L^2(\Omega) \to V_{\widetilde{\tau}}$. Let ϕ_i, $i = 1, 2, \dots, m$, be the nodal basis for $V_{\widetilde{\tau}}$. The operator $\widetilde{Q}_{\widetilde{\tau}}$ is given by

$$\widetilde{Q}_{\widetilde{\tau}} u = \sum_{i=1}^{m} \frac{(u, \phi_i)_{\widetilde{\tau}}}{(1, \phi_i)_{\widetilde{\tau}}} \phi_i, \tag{3.3}$$

with $(\cdot, \cdot)_{\widetilde{\tau}}$ the inner product in $L^2(\widetilde{\tau})$. For $u, v \in L^2(\widetilde{\tau})$,

$$(\widetilde{Q}_{\widetilde{\tau}} u, v)_{\widetilde{\tau}} = \sum_{i=1}^{m} \frac{(u, \phi_i)_{\widetilde{\tau}} (v, \phi_i)_{\widetilde{\tau}}}{(1, \phi_i)_{\widetilde{\tau}}}$$

and hence it immediately follows that $\widetilde{Q}_{\widetilde{\tau}}$ is symmetric on $L^2(\widetilde{\tau})$. Moreover, $\widetilde{Q}_{\widetilde{\tau}}$ is positive definite when restricted to $V_{\widetilde{\tau}}$ (see, Lemma 3.4). The next lemma provides a basic approximation property for $\widetilde{Q}_{\widetilde{\tau}}$.

Lemma 3.1. *Let τ be in $\mathcal{T}$. Then for $s = 0, 1$, there exists a constant C, independent of τ, such that*

$$\|u - \widetilde{Q}_{\widetilde{\tau}} u\|_{L^2(\widetilde{\tau})} \le C h_\tau^s \|u\|_{H^s(\widetilde{\tau})}, \quad \text{for all } u \in H^s(\widetilde{\tau}). \tag{3.4}$$

Proof. A simple computation shows that

$$\|\widetilde{Q}_{\widetilde{\tau}} u\|_{L^2(\widetilde{\tau})} \le C \|u\|_{L^2(\widetilde{\tau})}$$

from which (3.4) immediately follows for $s = 0$. For $s = 1$, let $\widetilde{u}$ be the constant function satisfying (3.1). Using the previous estimate, since $\widetilde{Q}_{\widetilde{\tau}} \widetilde{u} = \widetilde{u}$, we have

$$\|u - \widetilde{Q}_{\widetilde{\tau}} u\|_{L^2(\widetilde{\tau})} \le \|u - \widetilde{u}\|_{L^2(\widetilde{\tau})} + \|\widetilde{Q}_{\widetilde{\tau}}(u - \widetilde{u})\|_{L^2(\widetilde{\tau})} \le C \|u - \widetilde{u}\|_{L^2(\widetilde{\tau})}.$$

Combining the above inequalities and (3.1) completes the proof of (3.4) for $s = 1$. ☐

3.2. **Approximation properties: The multilevel case.** We provide some stronger approximation properties in the case when the mesh results from a multilevel refinement strategy. Again we describe the case of $d = 2$. The analogous constructions for $d > 2$, at least for the case of rectangular parallelepipeds, are straightforward generalizations. Assume that an initial coarse triangulation $\mathcal{T}_1$ of Ω has been provided with Γ_D aligning with the mesh $\mathcal{T}_1$. By this we mean that any edge of $\mathcal{T}_1$ on $\partial\Omega$ is either contained in Γ_D or intersects Γ_D at most at the endpoints of the edge. Multilevel triangulations are defined recursively. For $k > 1$, the triangulation $\mathcal{T}_k$ is defined by breaking each triangle in $\mathcal{T}_{k-1}$ into four, by connecting the centers of the edges. The finite element space V_k consists of the functions which are continuous on Ω, piecewise linear with respect to $\mathcal{T}_k$ and vanish on Γ_D. Let $h_k = \max_{\tau \in \mathcal{T}_k} h_\tau$. Clearly, $h_k = 2^{-k+1} h_1$.

We now define a sequence of approximation operators $\widetilde{Q}_k : L^2(\Omega) \to V_k$. Let ϕ_i, $i = 1, \dots, m$ be the nodal basis for V_k. We define $\widetilde{Q}_k$ by

$$\widetilde{Q}_k u = \sum_{i=1}^{m} \frac{(u, \phi_i)}{(1, \phi_i)} \phi_i. \tag{3.5}$$

Remark 3.1. *Let τ be a triangle of $\mathcal{T}_k$. It is easy to see that $\widetilde{Q}_{\tilde{\tau}} u$ and $\widetilde{Q}_k u$ agree on τ as long as $\tau \cap \Gamma_D = \emptyset$.*

In the multilevel case, we have the following stronger version of Lemma 3.1.

Lemma 3.2. *Let s be in $[0, 3/2)$. There exists a constant C_s not depending on h_k such that*

$$\|u - \widetilde{Q}_k u\|_{L^2(\Omega)} \le C_s h_k^s \|u\|_{H^s(\Omega)}, \qquad \text{for all } u \in H^s_D(\Omega).$$

For the proof of the lemma, we will use the following lemma which is a slight modification of Lemma 6.1 of [1]. Its proof is contained in the proof of Lemma 6.1 of [1].

Lemma 3.3. *Let Ω^η denote the strip $\{x \in \Omega \mid dist(x, \partial\Omega) < \eta\}$ and $0 \le s < 1/2$. Then for all $v \in H^{1+s}(\Omega)$,*

$$\|v\|_{H^1(\Omega^\eta)} \le C\eta^s \|v\|_{H^{1+s}(\Omega)}. \tag{3.6}$$

In addition, let Ω^η_D denote the strip $\{x \in \Omega \mid dist(x, \Gamma_D) < \eta\}$. Then for all v in $H^1_D(\Omega)$,

$$\|v\|_{L^2(\Omega^\eta_D)} \le C\eta \, \|v\|_{H^1(\Omega^\eta)}. \tag{3.7}$$

Proof of Lemma 3.2. The proof for $s = 0$ is trivial (see, Lemma 3.1). For positive s, we consider two cases. First we examine triangles whose boundaries do not intersect the boundary of any triangle in $\mathcal{T}_1$. We shall denote this set by $\tau \cap \mathcal{T}_1 = \emptyset$ and the remaining set of triangles by $\tau \cap \mathcal{T}_1 \neq \emptyset$.

Let ϕ_i be the nodal basis function in the space V_k associated with the node x_i^k. Assume that x_i^k does not lie on the boundary of any triangle $\tau \in \mathcal{T}_1$. Because of the multilevel construction, the mesh $\mathcal{T}_k$ is symmetric with respect to reflection through the point x_i^k. It follows that the nodal basis function ϕ_i, restricted to a line passing through x_i^k, is an even function with respect to x_i^k. Let $x_i^k = (p_1, p_2)$. Then, both of the functions $x - p_1$ and $y - p_2$ are odd on each such line. Consequently,

$$(x - p_1, \phi_i) = (y - p_2, \phi_i) = 0.$$

Thus, it follows from Remark 3.1 that $\widetilde{Q}_k \tilde{u}(x_i^k) = \tilde{u}(x_i^k)$ for any linear function $\tilde{u}$.

Let τ be a triangle whose boundary does not intersect the boundary of any triangle of $\mathcal{T}_1$. Applying the above argument to each node of τ shows that $\widetilde{Q}_k \tilde{u} = \tilde{u}$ on τ for any linear function $\tilde{u}$. Let $\tilde{\tau}$ be as in Lemma 3.1. Given $u \in H^2(\tilde{\tau})$, let $\tilde{u}$ be the linear function satisfying (3.2). As in the proof of Lemma 3.1, we get

$$\|u - \widetilde{Q}_k u\|_{L^2(\tau)} = \|u - \widetilde{Q}_{\tilde{\tau}} u\|_{L^2(\tau)} \le C\|u - \tilde{u}\|_{L^2(\tilde{\tau})} \le C h_k^s \|u\|_{H^s(\tilde{\tau})}$$

for $s = 0, 1, 2$. Summing the above inequality and interpolating gives

$$\left(\sum_{\tau \cap \mathcal{T}_1 = \emptyset} \|u - \widetilde{Q}_k u\|^2_{L^2(\tau)} \right)^{1/2} \le C h_k^s \|u\|_{H^s(\Omega)} \tag{3.8}$$

for $s \in [0, 2]$.

We next consider the case when τ intersects an edge in the triangulation $\mathcal{T}_1$. Suppose that τ intersects Γ_D. We clearly have that

$$\|\widetilde{Q}_k u\|_{L^2(\tau)} \le C\|u\|_{L^2(\widetilde{\tau})}. \tag{3.9}$$

Thus,

$$\|u - \widetilde{Q}_k u\|_{L^2(\tau)} \le C\|u\|_{L^2(\widetilde{\tau})}.$$

Summing the above inequality and applying (3.7) gives

$$\Big(\sum_{\tau\cap\Gamma_D\neq\emptyset} \|u - \widetilde{Q}_k u\|^2_{L^2(\tau)}\Big)^{1/2} \le Ch_k\|u\|_{H^1(\Omega^{2h_k})}. \tag{3.10}$$

Finally, we consider the case when τ intersects an edge in the triangulation $\mathcal{T}_1$ and does not intersect Γ_D. By Remark 3.1 and Lemma 3.1,

$$\|u - \widetilde{Q}_k u\|_{L^2(\tau)} \le Ch_k\|u\|_{H^1(\widetilde{\tau})}.$$

Summing the above inequality and using (3.10) gives

$$\Big(\sum_{\tau\cap\mathcal{T}_1\neq\emptyset} \|u - \widetilde{Q}_k u\|^2_{L^2(\tau)}\Big)^{1/2} \le Ch_k\|u\|_{H^1(E^{2h_k})}. \tag{3.11}$$

Here E^{2h_k} denotes the strip of width $\mathcal{O}(2h_k)$ around all element edges from the initial triangulation $\mathcal{T}_1$.

The lemma for $s = 1$ follows combining (3.8) and (3.11). The result for $s \in (0,1)$ follows by interpolation. For $1 < s < 3/2$, (3.6) and (3.11) imply

$$\Big(\sum_{\tau\cap\mathcal{T}_1\neq\emptyset} \|u - \widetilde{Q}_k u\|^2_{L^2(\tau)}\Big)^{1/2} \le Ch_k^s\|u\|_{H^s(\Omega)}.$$

The lemma for $1 < s < 3/2$ follows combining the above inequality with (3.8). This completes the proof of the lemma. □

Remark 3.2. *We can extend these arguments to the case when V_k consists of piecewise quadratic functions with respect to the k'th triangulation. Again $\{\phi_k\}$ denotes the nodal basis for V_k. Then $\widetilde{Q}_k$ defined by (3.5) satisfies Lemma 3.2. The proof is identical to the case of linears.*

3.3. **The coercivity estimate.** We next show that the coercivity estimate (2.4) holds for $\widetilde{Q}_k$. Actually, we only require that the triangulation $\mathcal{T}_h$ be locally quasi-uniform. We assume that Γ_D aligns with this triangulation and let V_h be the functions which are piecewise linear with respect to this triangulation, continuous on Ω and vanish on Γ_D. We consider the linear operator $\widetilde{Q}_h$ defined analogously to $\widetilde{Q}_k$ in (3.5) and show that

$$\|v\|^2 \le C(\widetilde{Q}_h v, v), \qquad \text{for all } v \in V_h.$$

The constant C above only depends on the quasi-uniformity constant (or minimal angle).

Let $\{x_i\}$ for $i = 1, \dots, m$ be the nodes of the triangulation and $\{\phi_i\}$ be the corresponding nodal basis functions. The mesh is quasi-uniform so for each ϕ_i, there is a parameter h_i such that

$$h_\tau \simeq h_i \tag{3.12}$$

holds for all triangles τ which have the node x_i as a vertex. Here we define $a \simeq b$ to mean that

$$a \le Cb \text{ and } b \le Ca$$

with constant C independent of the triangulation. It is well known that

$$(v,v) \simeq \sum_{\tau\in\mathcal{T}_h} h_\tau^2 \sum_{x_l\in\tau} v(x_l)^2, \qquad \text{for all } v \in V_h. \tag{3.13}$$

It follows from (3.12) that

$$(v,v) \simeq \sum_{i=1}^{m} h_i^2\, v(x_i)^2, \qquad \text{for all } v \in V_h. \tag{3.14}$$

We can now prove the coercivity estimate. This result was essentially given in [3] for the case of a globally quasi-uniform triangulation.

Lemma 3.4. *Assume that the mesh $\mathcal{T}_h$ is locally quasi-uniform. There is a constant C only depending on the quasi-uniformity condition such that*

$$C^{-1}(v,v) \le (\widetilde{Q}_h v, v) \le C\ (v,v), \qquad \textit{for all } v \in V_h.$$

Proof. Let G be the Gram matrix, i.e.

$$G_{ij} = (\phi_i, \phi_j), \quad i,j = 1,\dots,m$$

and D be the diagonal matrix with entries $D_{ii} = h_i^2$. Let v be in V_h and w be the coefficient vector satisfying

$$v = \sum_{i=1}^{m} w_i \phi_i.$$

Note that (3.14) can be rewritten as

$$C^{-1}((Gw,w)) \le ((Dw,w)) \le C((Gw,w)), \qquad \text{for all } w \in R^m.$$

Here $((\cdot,\cdot))$ denotes the inner product on R^m. This is equivalent to

$$C^{-1}((D^{-1}Gw,Gw)) \le ((Gw,w)) \le C((D^{-1}Gw,Gw)), \qquad \text{for all } w \in R^m.$$

Since

$$(1,\phi_i) \simeq h_i^2,$$

it follows that

$$\begin{aligned}(\widetilde{Q}_h v, v) &= \sum_{i=1}^{m} \frac{(v,\phi_i)^2}{(1,\phi_i)} = \sum_{i=1}^{m} \frac{((Gw)_i)^2}{(1,\phi_i)} \\ &\simeq ((D^{-1}Gw,Gw)) \simeq ((Gw,w)) = (v,v).\end{aligned}$$

This completes the proof of the lemma. □

4. Applications

In this section, we apply some of the above results. As we have seen in the previous section, the operator $\widetilde{Q}_k$ satisfies the approximation and coercivity estimates required for application of the abstract results. Throughout this section, we assume that $V_1 \subset V_2 \subset \dots$ is a sequence of nested piecewise linear and continuous multilevel spaces as described earlier. We take $V = L^2(\Omega)$ and $(\cdot,\cdot)$ to be the corresponding inner product. With a slight abuse of notation we also use $(\cdot,\cdot)$ to denote the obvious duality pairing.

Remark 4.1. *Since $V_k \subset H^s(\Omega)$, for $0 \le s < 3/2$, Q_k and $\widetilde{Q}_k$ extend naturally to all of $H^{-s}(\Omega)$. Let $-3/2 < s < 3/2$ and define $\mathcal{A}^s$ as in (1.2). It is known that the norm $(\mathcal{A}^s u, u)^{1/2}$ is equivalent to $\|\cdot\|_{H^s(\Omega)}$; cf.* [14].

Fix $\gamma < 3/2$. By Lemma 3.2, the triangle inequality and well known properties of Q_k

$$\|(Q_k - \widetilde{Q}_k)u\|_{L^2(\Omega)} \le C\theta_k^{-1/2}\|u\|_{H^\gamma(\Omega)}$$

where $\theta_k = h_k^{-2\gamma}$. Let $s < \gamma$ and set $\mu_k = h_k^{-2s}$. Then,

$$\ell_{k,j} = \left(\frac{h_k}{h_j}\right)^{\gamma - s}$$

decays exponentially as a function of $k - j$. An elementary computation gives that

$$\|\mathcal{L}\| \le C_{\mathcal{L}} = \left(1 - \left(\frac{1}{2}\right)^{\gamma - s}\right)^{-1}.$$

The next theorem immediately follows from Remark 4.1, Remark 2.2 and Theorem 2.1.

Theorem 4.1. *Let $-3/2 < s < 3/2$. Then $(\widetilde{\mathcal{A}}^{(s)}u, u)^{1/2}$ provides a norm on $H^s_D(\Omega)$ which is equivalent to the usual Sobolev norm. Here*

$$\widetilde{\mathcal{A}}^{(s)}u = \sum_{k=1}^{\infty} h_k^{-2s}(\widetilde{Q}_k - \widetilde{Q}_{k-1})^2 u.$$

4.1. A preconditioning example.

We consider applying the earlier results to develop a preconditioner for an example involving a pseudo-differential operator of order minus one. The canonical example of such an application is associated with a form

$$\mathcal{V}(u, v) = \int_\Omega \int_\Omega \frac{u(s_1)v(s_2)}{|s_1 - s_2|}\, ds_1 ds_2.$$

For this application, Γ_D is empty and we seek preconditioners for the problem: Find $U \in V_J$ satisfying

$$\mathcal{V}(U, \phi) = F(\phi) \quad \text{for all } \phi \in V_J.$$

Here F is a given functional. It is shown in [2] that

$$\mathcal{V}(u, u) \simeq \|u\|^2_{H^{-1/2}(\Omega)} \quad \text{for all } u \in V_J. \tag{4.1}$$

It is convenient to consider the problem of preconditioning in terms of operators. Specifically, let $\mathcal{V} : V_J \to V_J$ be defined by

$$(\mathcal{V}v, w) = \mathcal{V}(v, w) \quad \text{for all } v, w \in V_J.$$

We shall see that $\widetilde{\mathcal{A}}_J^{(1/2)}$ defined by

$$\widetilde{\mathcal{A}}_J^{(1/2)} = \sum_{k=1}^{J} h_k^{-1}(\widetilde{Q}_k - \widetilde{Q}_{k-1})^2$$

provides a computationally efficient preconditioner for $\mathcal{V}$. Indeed, by Theorem 2.1,

$$(\widetilde{\mathcal{A}}_J^{(1/2)}u, u) \simeq (\mathcal{A}^{1/2}u, u) \quad \text{for all } u \in V_J.$$

Applying Remark 4.1 and (4.1) implies that

$$(\mathcal{V}\mathcal{A}^{1/2}u, \mathcal{A}^{1/2}u) \simeq (\mathcal{A}^{-1/2}\mathcal{A}^{1/2}u, \mathcal{A}^{1/2}u) = (u, \mathcal{A}^{1/2}u)$$

for all $u \in V_J$. Thus, $\widetilde{\mathcal{A}}_J^{(1/2)}\mathcal{V}$ has a bounded spectral condition number.

It is easy to evaluate the action of $\widetilde{\mathcal{A}}_J^{(1/2)}$ in a preconditioned iteration procedure. For $k = 1, 2, \ldots, J$, let $\{\phi_i^k\}$ denote the nodal basis for V_k. In typical preconditioning applications, one is required to evaluate the action of the preconditioner on a function v where only the quantities $\{(v, \phi_i^J)\}$ are known. One could, of course, compute v from $\{(v, \phi_i^J)\}$ but this would require solving a Gram matrix problem. Our preconditioner avoids the Gram matrix problem. To evaluate the action of $\widetilde{Q}_k$, for $1 \le k \le J$, one is only required to take linear combinations of the quantities $\{(v, \phi_i^k)\}$. Note that (v, ϕ_i^k) is a simple linear combination of $\{(v, \phi_i^{k+1})\}$. Thus, we see that all of the $\widetilde{Q}_k$'s can be computed efficiently (with work proportional to the number of unknowns on the finest level J) by a V-cycle-like algorithm.

4.2. **Examples involving sums of operators.**

We next consider preconditioning a sum of operators. The first example involves preconditioning the discrete systems which result from time stepping a parabolic initial value problem. The second example considers a Tikhonov regularization of a problem with noisy data.

Fully discrete time stepping schemes for parabolic problems often lead to problems of the form: Find $u \in S_h$ satisfying

$$(u, \phi) + \epsilon D(u, \phi) = F(\phi) \quad \text{for all } \phi \in S_h. \tag{4.2}$$

Here $D(\cdot, \cdot)$ denotes the Dirichlet form on Ω and S_h is the finite element approximation. The parameter ϵ is related to the time step size and is often small. Assume that $S_h = V_J$ where V_J is a multilevel approximation space as developed earlier. Let $\mu_k = 1$ and $\hat{\mu}_k = h_k^2$, for $k = 1, 2, \ldots$. For convenience, we assume that Γ_D is non-empty so that $D(v, v) \simeq \|v\|_1^2$, for all $v \in H_D^1(\Omega)$. Then for L_J and $\hat{L}_J$ defined respectively by (2.16) and (2.18), we have

$$(L_J v, v) \simeq (v, v) \text{ and } (\hat{L}_J v, v) \simeq D(v, v)$$

for all $v \in V_J$. Applying Corollary 2.1 gives that

$$B_J = \sum_{k=1}^{J} (\mu_k^{-1} + \epsilon\hat{\mu}_k^{-1})^{-1}(\widetilde{Q}_k - \widetilde{Q}_{k-1})^2 \tag{4.3}$$

provides a uniform preconditioner for the discrete operator associated with (4.2). The resulting condition number for the preconditioned system can be bounded independently of the time step size ϵ and the number of levels J.

We next consider an example which results from Tikhonov regularization of a problem with noisy data. We consider approximating the solution of the problem

$$Tv = f$$

where T denotes the inverse of the Laplacian and $f \in L^2(\Omega)$. This is replaced by the discrete problem

$$T_h v = f_h$$

where T_h is the Galerkin solution operator, i.e., $T_h v = w$ where $w \in V_J$ satisfies

$$D(w, \theta) = (v, \theta) \quad \text{for all } \theta \in V_J$$

and f_h is the $L^2(\Omega)$ orthogonal projection onto V_J. If it is known that v is smooth but f is noisy, better approximations result from regularization [11], [17]. We consider the regularized solution $\widetilde{w} \in V_J$ satisfying

$$(T_h + \alpha A_h)\widetilde{w} = f_h. \tag{4.4}$$

Here $A_h : V_J \to V_J$ is defined by

$$(A_h v, w) = D(v, w) \quad \text{for all } v, w \in V_J.$$

The regularization parameter α is often small (see, [17]) and can be chosen optimally in terms of the magnitude of the noise in f.

Preconditioners for the sum in (4.4) of the form of (4.3) result from the application of Corollary 2.1. In this case, $\mu_k = h_h^{-2}$, $\hat{\mu}_k = h_k^2$. The condition numbers for the resulting preconditioned systems can be bounded independent of the regularization parameter α.

Preconditioners for systems like (4.4) are generally not easily developed. The problem is that the operator applied to the higher frequencies (depending on the size of α) behaves like a differential operator while on the lower frequencies, it behaves like the inverse of a differential operator. This causes difficulty in most multilevel methods.

4.3. $H^1(\Omega)$ **bounded extensions.** As a final application, we consider the construction of $H^1(\Omega)$ bounded extensions. Such extensions are useful in development of domain decomposition preconditioners with inexact subdomain solves. The construction given here is essentially the same as that in [12]. We include it here in detail as an application of Theorem 4.1.

With $\{V_j\}$ as above, let $\widetilde{V}_k$ (for $k = 1, 2, \ldots, J$) be the functions defined on $\partial\Omega$ which are restrictions of those in V_k. This gives a multilevel structure on the finest space $\widetilde{V}_J$. These spaces inherit a nodal basis from the original nodal basis on V_k. The nodal basis function associated with a boundary node x_i is just the restriction of the basis function for V_k associated with x_i. Denoting this basis by $\{\psi_i^k\}$, we define

$$\widetilde{q}_k(f) = \sum \frac{< f, \psi_i^k >}{< 1, \psi_i^k >} \psi_i^k.$$

The above sum is taken over the nodal basis elements for $\widetilde{V}_k$ and $< \cdot, \cdot >$ denotes the $L^2(\partial\Omega)$ inner product. We note that it is known [14] that

$$\|\theta\|_{H^{1/2}(\partial\Omega)}^2 \simeq \sum_{k=1}^{J} h_k^{-1} \|(q_k - q_{k-1})\theta\|_{L^2(\partial\Omega)}^2$$

where q_k denotes the L^2-projection onto $\tilde{V}_k$. It is easy to see that Theorem 4.1 holds for these spaces. Thus

$$\|\theta\|^2_{H^{1/2}(\partial\Omega)} \simeq \sum_{k=1}^{J} h_k^{-1} \|(\tilde{q}_k - \tilde{q}_{k-1})\theta\|^2_{L^2(\partial\Omega)}, \tag{4.5}$$

with $\tilde{q}_J\theta = \theta$ and $\tilde{q}_0\theta = 0$.

Now given a function $\theta \in \tilde{V}_J$, we define $E_J\theta \in V_J$ by $E_J\theta = \sum_{k=1}^J \omega_k$ with ω_k defined as follows. Let $\bar{\theta}$ be the mean value of θ on $\partial\Omega$. Then ω_1 is the function in V_1 satisfying

$$\omega_1(x_i) = \begin{cases} \tilde{q}_1(x_i) & \text{if } x_i \text{ is a node of } V_1 \text{ on } \partial\Omega, \\ \bar{\theta} & \text{if } x_i \text{ is a node of } V_1 \text{ in the interior of } \Omega. \end{cases}$$

For $J \ge k > 1$, ω_k is the function in V_k satisfying

$$\omega_k(x_i) = \begin{cases} [\tilde{q}_k\theta - \tilde{q}_{k-1}\theta](x_i) & \text{if } x_i \text{ is a node of } V_k \text{ on } \partial\Omega, \\ 0 & \text{if } x_i \text{ is a node of } V_k \text{ in the interior of } \Omega. \end{cases}$$

Note that $E_J\theta = \theta$ on $\partial\Omega$ so that E_J is an extension operator.

Recall that $|\cdot|_{H^1(\Omega)}$ denotes the semi-norm on $H^1(\Omega)$. Then

$$|E_J\theta|_{H^1(\Omega)} = |E_J\theta - \bar{\theta}|_{H^1(\Omega)} = |E_J(\theta - \bar{\theta})|_{H^1(\Omega)} \le \|E_J(\theta - \bar{\theta})\|_{H^1(\Omega)}.$$

We now use the following well known multilevel characterization of the $H^1(\Omega)$ norm on V_J:

$$\|v\|^2_{H^1(\Omega)} \simeq \inf \sum_{k=1}^{J} h_k^{-2} \|v_k\|^2_{L^2(\Omega)},$$

where the infimum is taken over all splittings $v = \sum_{k=1}^J v_k$, with $v_k \in V_k$. Applying this with $v = E_J(\theta - \bar{\theta}) = (\omega_1 - \bar{\theta}) + \sum_{k=2}^J \omega_k$ and using (4.5), we conclude that

$$\begin{aligned} \|E_J(\theta - \bar{\theta})\|^2_{H^1(\Omega)} &\le C\left[\sum_{k=2}^{J} h_k^{-2}\|\omega_k\|^2_{L^2(\Omega)} + h_1^{-2}\|\omega_1 - \bar{\theta}\|^2_{L^2(\Omega)}\right] \\ &\le C\sum_{k=1}^{J} h_k^{-1}\|(\tilde{q}_k - \tilde{q}_{k-1})(\theta - \bar{\theta})\|^2_{L^2(\partial\Omega)} \le C\|\theta - \bar{\theta}\|^2_{H^{1/2}(\partial\Omega)} \le C|\theta|^2_{H^{1/2}(\partial\Omega)}, \end{aligned}$$

where $|\cdot|_{H^{1/2}(\partial\Omega)}$ denotes the $H^{1/2}(\partial\Omega)$ semi-norm. Thus we see that

$$|E_J\theta|_{H^1(\Omega)} \le C|\theta|_{H^{1/2}(\partial\Omega)}.$$

This type of bounded extension operator is precisely what is required for the development of non-overlapping domain decomposition algorithms with inexact solves.

5. NUMERICAL RESULTS

We present the results of some numerical experiments using the operator $\tilde{\mathcal{A}}_J$ defined by (1.4) applied as a preconditioner for various discrete differential and pseudo-differential operators. The first example is a standard one and involves preconditioning the finite element discretization of a second order problem. Although there are many methods available for this problem, we consider it here since it is the most well studied problem. The second problem considered involves using (4.3) to precondition a sum of operators similar to (4.4).

We start with preconditioning the Laplace operator with Neumann boundary conditions. To make this problem definite, we consider both the finite element operator and the preconditioner on the L^2-orthogonal complement of the one dimensional subspace of constants. The finite element space $\widetilde{V}_k$ consists of piecewise linear functions defined with respect to a uniform triangulation of the square which results when an equally spaced $n_k \times n_k$ mesh of smaller squares is partitioned into triangles by connecting the lower left and upper right hand vertices. Here we take $n_k = 2^k$ and define $\widetilde{V}_k$ to be the functions in V_k which are orthogonal to constants. Let $\widetilde{Q}_k$ be defined as in (3.5) with respect to the space $\widetilde{V}_k$.

The BPX-like preconditioner

$$\widetilde{B}_J = \sum_{k=0}^{J} h_k^2 \widetilde{Q}_k$$

provides a uniform preconditioner for the Galerkin discretization of the problem

$$\begin{aligned} u - \Delta u &= f \quad \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

using the approximation subspace $\widetilde{V}_J$. Here n denotes the outward normal direction on $\partial\Omega$. Let Q_J denote the $L^2(\Omega)$-orthogonal projector onto V_J. The operator $Q_J \widetilde{B}_J$ is symmetric and positive definite on V_J and is a uniform preconditioner for the Galerkin approximation A_h to

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{5.1}$$

based on the approximation subspace V_J.

Table 1 reports the condition number of $Q_J \widetilde{B}_J$ as a function of h_J for both the BPX preconditioner and the preconditioner defined by

$$B_J = \sum_{k=2}^{J} h_k^2 (\widetilde{Q}_k - \widetilde{Q}_{k-1})^2.$$

Note that B_J annihilates constants since we have omitted the $k = 1$ term in the above sum. We see that in this simple case, the new preconditioner is somewhat better than the BPX-like preconditioner although it is slightly more complicated to apply.

Table 1. Condition numbers for BPX and B_J applied to (5.1).

h_J	K(BPX)	$K(B_J)$
1/8	9.8	5.9
1/16	11.3	7.3
1/32	12.1	8.4
1/64	12.9	9.1
1/128	13.4	9.6
1/256	13.7	10.1
1/512	13.9	10.4

The second example illustrates the performance of a preconditioner of the form of (2.19) applied to the problem $\alpha A_h + T_h$. Specifically,

$$B_J = \sum_{k=2}^{J} (\alpha h_k^{-2} + h_k^2)^{-1} (\widetilde{Q}_k - \widetilde{Q}_{k-1})^2. \tag{5.2}$$

The operator T_h used here is an operator which is spectrally very close to A_h^{-1}. It is defined to be the solution operator of the problem: Find $w \in V_J$ satisfying

$$D(w, \phi) = (v, \phi)_* \quad \text{for all } \phi \in V_J,$$

i.e., $T_h v = w$. The inner product $(\cdot,\cdot)_*$ is a minor perturbation of $(\cdot,\cdot)$ which makes the computation of T_h feasible via the fast Fourier transform.

We report the condition numbers for h_J between 1/8 and 1/512 and for $\alpha = h_J^\gamma$, $\gamma = 0, 1, 2, 4$. Although there are some values of γ for which $\alpha A_h + T_h$ can be preconditioned by other methods, (5.2) provides good preconditioning for all choices of the parameter as guaranteed by Corollary 2.1. Note in particular the examples $\gamma = 1$ and $\gamma = 2$. For these cases, the operator behaves like a differential operator on higher frequencies and like a pseudo-differential operator of negative order on the lower frequencies. As far as we know, methods for preconditioning such an operator are not available in the literature.

TABLE 2. Condition numbers when preconditioning $h_J^\gamma A_h + T_h$.

h_J	$\gamma = 0$	$\gamma = 1$	$\gamma = 2$	$\gamma = 4$
1/8	5.9	12.0	21.6	14.9
1/16	7.2	17.6	35.8	15.5
1/32	8.1	24.3	43.2	16.4
1/64	8.9	34.2	46.8	16.9
1/128	9.5	45.9	50.8	17.1
1/256	10.0	55.2	54.5	17.3
1/512	10.4	61.1	57.6	17.4

REFERENCES

[1] J. H. Bramble and J. E. Pasciak New estimates for multigrid algorithms including the V-cycle *Math. Comp.* 60(1993),447–471.

[2] J.H. Bramble, Z. Leyk, and J.E. Pasciak, Iterative schemes for non-symmetric and indefinite elliptic boundary value problems,*Math. Comp.* 60 (1993) 1–22.

[3] J. H. Bramble, J. E. Pasciak and J. Xu, Parallel multilevel preconditioners, *Math. Comp.* 55(1990), 1–22.

[4] J. H. Bramble and P. S. Vassilevski, Wavelet–like extension operators in interface domain decomposition methods, (unpublished manuscript) 1997.

[5] J. M. Carnicer, W. Dahmen, and J. M. Peña, Local decompositions of refinable spaces and wavelets, *Appl. Comp. Harm. Anal.* 3(1996), 127–153.

[6] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, *North-Holland*,New York, 1978.

[7] P. Grisvard, Elliptic Problems in Nonsmooth Domains, *Pitman*, Boston, 1985.

[8] G. Haase, U. Langer, A. Meyer, and S. V. Nepomnyaschikh, Hierarchical extension operators and local multigrid methods in domain decomposition preconditioners, *East–West J. Numer. Math.* 2(1994), 173–193.

[9] U. Kotyczka and P. Oswald, Piecewise linear prewavelets of small support, in: *Approximation Theory VIII*, vol. 2 (wavelets and multilevel approximation), C. Chui, L.L. Schumaker, eds., World Scientific, Singapore, 1995, 235-242.

[10] R. Lorentz and P. Oswald, Constructing economical Riesz bases for Sobolev spaces, *Proceedings of the Domain Decomposition Conference held in Bergen, Norway, June 3–8, 1996.*

[11] F. Natterer, Error bounds for Tikhonov regularization in Hilbert scales, *Appl. Anal.* 18(1984), 29–37.

[12] S. V. Nepomnyaschikh, Optimal multilevel extension operators, *Report SPC 95–3, Jan, 1995*, Technische Universität Chemnitz–Zwickau, Germany.

[13] P. Oswald, On discrete norm estimates related to multilevel preconditioners in the finite element method, Constructive Theory of Functions, Proc. Int. Conf. Varna 1992, *Bulg. Acad. Sci.*, Sophia, 1992, 203–214.

[14] P. Oswald, Multilevel Finite Element Approximation, Theory and Applications *B.G. Teubner* Stuttgart, Germany 1994.

[15] R. Stevenson, A robust hierarchical basis preconditioner on general meshes, Report # 9533, *Department of Mathematics, University of Nijmegen, Nijmegen, The Netherlands*, 1995.

[16] R. Stevenson, Piecewise linear (pre-) wavelets on non–uniform meshes, Report # 9701, *Department of Mathematics, University of Nijmegen, Nijmegen, The Netherlands*, 1997.

[17] U. Tautenhahn, Error estimates for regularization methods in Hilbert scales, *SIAM J. Numer. Anal.* 33(1996), 2120–2130.

[18] P. S. Vassilevski and J. Wang, Stabilizing the hierarchical basis by approximate wavelets, I: Theory, *Numer. Linear Alg. Appl.* 4(1997), 103–126.

[19] P. S. Vassilevski and J. Wang, Stabilizing the hierarchical basis by approximate wavelets, II: Implementation and numerical experiments, *SIAM J. Sci. Comput.*, to appear.

[20] X. Zhang, Multi-level Additive Schwarz Methods, *Courant Inst. Math. Sci., Dept. Comp. Sci. Rep.* 1991.

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