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where $F \subset \partial\Omega_i$ is any face of Ω_i and $P_F : L^2(F) \mapsto S_h(F)$ is the orthogonal $L^2(F)$ projection.

$$\begin{aligned}
\|w_i - w\|_{L^2(\Omega_i)}^2 &\lesssim h^3 \sum_{x \in \partial\Omega_i} (w_i - w)^2(x) \\
&\lesssim h^3 \sum_{F \subset \partial\Omega_i} \sum_{x \in F} (w_i - w)^2(x) \\
&\lesssim h^3 \sum_{F \subset \partial\Omega_i} \left(\sum_{x \in F} (w_i - P_F u)^2(x) + \sum_{x \in \partial F} w_i^2(x) \right) \\
&\lesssim \sum_{F \subset \partial\Omega_i} (h \|w_i - P_F u\|_{L^2(F)}^2 + h^2 \|w_i\|_{L^2(\partial F)}^2).
\end{aligned}$$

Again we need to bound two terms appearing in the last expression above. For the first term, we have

$$\begin{aligned}
\sum_{F \subset \partial\Omega_i} h \|w_i - P_F u\|_{L^2(F)}^2 &\lesssim h \|u - w_i\|_{L^2(\partial\Omega_i)}^2 \\
&\lesssim \|u - w_i\|_{L^2(\Omega_i)}^2 + h^2 \|u - w_i\|_{H^1(\partial\Omega_i)}^2 \\
&\lesssim h^2 \|u\|_{H^1(\Omega_i)}^2
\end{aligned}$$

where we have used Lemma 2.1 and (4.5).

The second term can be bounded by the discrete Sobolev inequality in Lemma 2.4

$$\sum_{F \subset \partial\Omega_i} h^2 \|w_i\|_{L^2(\partial F)}^2 \lesssim h^2 |\log h| \|u\|_{H^1(\Omega_i)}^2 \lesssim h^2 |\log h| \|u\|_{H^1(\Omega_i)}^2$$

Consequently

$$\|w - w_i\|_{L^2(\Omega_i)} \lesssim h |\log h|^{\frac{1}{2}} \|u\|_{H^1(\Omega_i)}.$$

As in (4.9), the estimate for $d = 3$ follows. This completes the proof. \square

As a direct consequence of the above lemma, we have

THEOREM 4.7. *If for all i , the $d - 1$ dimensional Lebesgue measure of $(\partial\Omega_i \cap \partial\Omega)$ is positive, then for all $u \in H_0^1(\Omega)$*

$$(4.10) \quad \|(I - Q_h^\omega)u\|_{L_\omega^2(\Omega)} \lesssim h |\log h|^{\frac{1}{2}} |u|_{H_\omega^1(\Omega)},$$

and

$$(4.11) \quad |Q_h^\omega u|_{H_\omega^1(\Omega)} \lesssim |\log h|^{\frac{1}{2}} |u|_{H_\omega^1(\Omega)}.$$

Proof. By hypothesis, we have $\|u\|_{H_\omega^1(\Omega)} \lesssim |u|_{H_\omega^1(\Omega)}$ by Poincaré inequality. The estimate (4.10) then follows from (4.8). The estimate (4.11) can be proved similarly as (3.11) by using (4.10). \square

(4.12) Remark. 4.1 The assumption concerning the measure of $meas_{d-1}(\partial\Omega_i \cap \partial\Omega)$ in Theorem 4.7 cannot be removed in general and the deterioration $\underline{h}^{-\frac{1}{2}}$ in the estimate of Theorem 4.5 is also best possible. All these issues will be discussed in a separate paper.

Proof. The proof will be carried out separately for different dimensions even though the ideas in both cases are quite similar.

Case 1: $d = 2$. Let w_i be as in the proof of Lemma 4.2 and define $w \in S_h$ by

$$w = \begin{cases} w_i, & \text{at the nodes inside } \Omega_i; \\ P_e u, & \text{at the nodes inside } e \subset \partial\Omega_i; \\ 0, & \text{at nodes elsewhere.} \end{cases}$$

where $e \subset \partial\Omega_i$ is any edge of Ω_i and $P_e : L^2(e) \mapsto S_h(e)$ is the orthogonal $L^2(e)$ projection.

By (2.2)

$$\begin{aligned} \|w_i - w\|_{L^2(\Omega_i)}^2 &\lesssim h^2 \sum_{x \in \partial\Omega_i} (w_i - w)^2(x) \\ &\lesssim h^2 \sum_{e \subset \partial\Omega_i} \sum_{x \in e} (w_i - w)^2(x) \\ &\lesssim h^2 \sum_{e \subset \partial\Omega_i} \left(\sum_{x \in e} (w_i - P_e u)^2(x) + \sum_{x \in \partial e} w_i^2(x) \right). \end{aligned}$$

We need to bound the two terms appearing in the above last expression. The first term is easy:

$$\begin{aligned} \sum_{e \subset \partial\Omega_i} h \|w_i - P_e u\|_{L^2(e)}^2 &\lesssim h \|u - w_i\|_{L^2(\partial\Omega_i)}^2 \\ &\lesssim \|u - w_i\|_{L^2(\Omega_i)}^2 + h^2 \|u - w_i\|_{H^1(\partial\Omega_i)}^2 \\ &\lesssim h^2 \|u\|_{H^1(\Omega_i)}^2 \end{aligned}$$

where we have used Lemma 2.1 and (4.5).

The second term can be bounded by the second discrete Sobolev inequality in Lemma 2.3.

$$\sum_{e \subset \partial\Omega_i} h^2 \sum_{x \in \partial e} w_i^2(x) \lesssim h^2 |\log h| \|w_i\|_{H^1(\Omega_i)}^2 \lesssim h^2 |\log h| \|u\|_{H^1(\Omega_i)}^2.$$

Consequently

$$\|w - w_i\|_{L^2(\Omega_i)} \lesssim h |\log h|^{\frac{1}{2}} \|u\|_{H^1(\Omega_i)}.$$

Applying the above estimate, with the triangle inequality and (4.5), we get

$$(4.9) \quad \|u - w\|_{L^2(\Omega_i)} \leq \|u - w_i\|_{L^2(\Omega_i)} + \|w_i - w\|_{L^2(\Omega_i)} \lesssim h |\log h|^{\frac{1}{2}} \|u\|_{H^1(\Omega_i)}.$$

The desired result for $d = 2$ then follows.

Case 2: $d = 3$. Again, let w_i be as in the proof of Lemma 4.2 and define $w \in S_h$ by

$$w = \begin{cases} w_i, & \text{at the nodes inside } \Omega_i; \\ P_F u, & \text{at the nodes inside } F \subset \partial\Omega_i; \\ 0, & \text{at nodes elsewhere} \end{cases}$$

LEMMA 4.4. For any $u \in S_{\underline{h}}$ and $\tau \in \mathcal{T}_h$,

$$\|(I - I_h)u\|_{L^2(\tau)} \lesssim \begin{cases} h(\log \frac{h}{\underline{h}})^{\frac{1}{2}}|u|_{H^1(\tau)} & \text{if } d = 2 \\ h(\frac{h}{\underline{h}})^{\frac{1}{2}}|u|_{H^1(\tau)} & \text{if } d = 3 \end{cases}$$

where $I_h : S_{\underline{h}} \mapsto S_h$ is the interpolation operator.

Proof. We first consider the case that $d = 2$. Let $\hat{\tau}$ be the standard reference element, then

$$\|(I - I_h)u\|_{L^2(\tau)} \lesssim h\|(\hat{I} - \hat{I}_h)\hat{u}\|_{L^2(\hat{\tau})}.$$

It follows from the discrete Sobolev inequality in Lemma 2.3 for $d = 2$ that

$$\|(\hat{I} - \hat{I}_h)\hat{u}\|_{L^2(\hat{\tau})} \leq 2\|\hat{u}\|_{L^\infty(\hat{\tau})} \lesssim (\log \frac{h}{\underline{h}})^{\frac{1}{2}}\|\hat{u}\|_{H^1(\hat{\tau})}.$$

Replacing \hat{u} by $\hat{u} + c$ for any constant c , we have

$$\begin{aligned} \|(\hat{I} - \hat{I}_h)\hat{u}\|_{L^2(\hat{\tau})} &\lesssim (\log \frac{h}{\underline{h}})^{\frac{1}{2}} \inf_{c \in \mathbb{R}^1} \|\hat{u} + c\|_{H^1(\hat{\tau})} \\ &\lesssim (\log \frac{h}{\underline{h}})^{\frac{1}{2}}|\hat{u}|_{H^1(\hat{\tau})} \lesssim (\log \frac{h}{\underline{h}})^{\frac{1}{2}}|u|_{H^1(\tau)}. \end{aligned}$$

The desired result for $d = 2$ then follows. The proof for $d = 3$ only differs in the type of Sobolev inequality (in (2.1)) used and the details obviously need not be repeated here. This completes the proof. \square

As a direct consequence of Lemma 4.4, we have

THEOREM 4.5. For any $u \in S_{\underline{h}}$

$$\|(I - Q_h^\omega)u\|_{L_\omega^2(\Omega)} \lesssim \begin{cases} h(\log \frac{h}{\underline{h}})^{\frac{1}{2}}|u|_{H_\omega^1(\Omega)}, & \text{if } d = 2; \\ h(\frac{h}{\underline{h}})^{\frac{1}{2}}|u|_{H_\omega^1(\Omega)}, & \text{if } d = 3 \end{cases}$$

and

$$|Q_h^\omega u|_{H_\omega^1(\Omega)} \lesssim \begin{cases} (\log \frac{h}{\underline{h}})^{\frac{1}{2}}|u|_{H_\omega^1(\Omega)}, & \text{if } d = 2; \\ (\frac{h}{\underline{h}})^{\frac{1}{2}}|u|_{H_\omega^1(\Omega)}, & \text{if } d = 3. \end{cases}$$

4.2.2. Estimates for General H^1 Functions. In this subsection, we shall derive some estimates for functions in H^1 .

The following lemma shows that nearly optimal estimates can be obtained in general if the full weighted H^1 norms are used.

LEMMA 4.6. For all $u \in H_0^1(\Omega)$

$$(4.8) \quad \|(I - Q_h^\omega)u\|_{L_\omega^2(\Omega)} \lesssim h|\log h|^{\frac{1}{2}}\|u\|_{H_\omega^1(\Omega)}.$$

THEOREM 4.3. *Assume the decomposition (4.1) has no internal cross points. Then, for all $u \in H_0^1(\Omega)$*

$$(4.6) \quad \|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \lesssim h|u|_{H^1_\omega(\Omega)},$$

and

$$(4.7) \quad |Q_h^\omega u|_{H^1_\omega(\Omega)} \lesssim |u|_{H^1_\omega(\Omega)}.$$

Proof. Define a function $\phi \in S_h(\Gamma)$ by

$$\phi = P_{\Gamma_i}^k u, \quad \text{on each } \Gamma_i$$

where $P_{\Gamma_i}^k : L^2(\Gamma_i) \mapsto \mathcal{M}(\Gamma_i)$ is the orthogonal L^2 -projection. By the hypothesis that Γ has no cross point, ϕ is well defined. Note that on each Γ_i , we have

$$\|u - \phi\|_{L^2(\Gamma_i)} \leq \|u - w_i\|_{L^2(\Gamma_i)}.$$

Due to Lemma 2.1

$$\|u - w_i\|_{L^2(\Gamma_i)}^2 \leq h^{-1} \|u - w_i\|_{L^2(\Omega_i)}^2 + h \|u - w_i\|_{H^1(\Omega_i)}^2.$$

Hence,

$$\begin{aligned} h \|u - w_i\|_{L^2(\Gamma_i)}^2 &\lesssim \|u - w_i\|_{L^2(\Omega_i)}^2 + h^2 \|u - w_i\|_{H^1(\Omega_i)}^2 \\ &\lesssim h^2 |u|_{H^1(\Omega_i)}^2. \end{aligned}$$

Consequently,

$$\begin{aligned} h \|u - P_\Gamma^k u\|_{L^2_\omega(\Omega)}^2 &\lesssim h \sum_{i=1}^J \omega_i \|u - \phi\|_{L^2(\Gamma_i)}^2 \\ &\lesssim h^2 \sum_{i=1}^J \omega_i |u|_{H^1(\Omega_i)}^2 = h^2 |u|_{H^1_\omega(\Omega)}^2. \end{aligned}$$

Applying Lemma 4.2 gives to (4.6).

The proof of (4.7) is identical to that of (3.11). This completes the proof. \square

4.2. General Case. When the interface has some internal cross points, the problem becomes somewhat more subtle. We will derive certain estimates under some special circumstances.

4.2.1. Estimates for “finer” finite element functions. For $d = 2$, the embedding $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$ is not true in general but it is “almost right” for the functions in finite element subspaces as it is indicated by the second inequality in Lemma 2.3. This observation is the main motivation for the result in this paragraph and the argument is similar to that used in the proof of Proposition 4.1.

4.1. No Internal Cross Point Case. What we mean by internal cross points are those points on Γ that belong to more than two $\bar{\Omega}_i$'s. If there is no such point on the interface, the analysis becomes very simple and optimal estimates can be derived.

We shall first present a lemma that shows that the estimate we need can be reduced to the estimates on interfaces. To do this, let us introduce a weighted inner product on $L^2(\Gamma)$

$$(u, v)_{L^2_\omega(\Gamma)} = \sum_{i=1}^J \int_{\partial\Omega_i \setminus \partial\Omega} \omega_i u v dx.$$

Denoting $S_h(\Gamma) \stackrel{def}{=} \{v|_\Gamma : v \in S_h\}$, let $P_\Gamma^k : L^2(\Gamma) \mapsto S_h(\Gamma)$ be the orthogonal projection with respect to $(\cdot, \cdot)_{L^2_\omega(\Gamma)}$.

LEMMA 4.2. *For all $u \in H_0^1(\Omega)$*

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \lesssim h|u|_{H^1_\omega(\Omega)} + h^{1/2}\|u - P_\Gamma^k u\|_{L^2_\omega(\Gamma)}.$$

Proof. On each domain Ω_i , by Proposition 3.5, there exists a $w_i \in S_h(\Omega_i)$ such that

$$(4.5) \quad \|u - w_i\|_{L^2(\Omega_i)}^2 + h^2\|u - w_i\|_{H^1(\Omega_i)}^2 \lesssim h^2\|u\|_{H^1(\Omega_i)}^2.$$

Let $w \in S_h$ be such that

$$w = \begin{cases} w_i, & \text{at the nodes in } \Omega_i; \\ P_\Gamma^k u, & \text{on } \Gamma. \end{cases}$$

Therefore, using (2.2)

$$\begin{aligned} \|u - w\|_{L^2_\omega(\Omega)}^2 &\lesssim \sum_{i=1}^J \omega_i \|u - w_i\|_{L^2(\Omega_i)}^2 + h^2 \sum_{i=1}^J \omega_i \sum_{p \in \Gamma_i} |(w - P_\Gamma^k u)(p)|^2 \\ &\lesssim \sum_{i=1}^J \omega_i \|u - w_i\|_{L^2(\Omega_i)}^2 + h \|w - P_\Gamma^k u\|_{L^2_\omega(\Gamma)}^2 \\ &\lesssim \sum_{i=1}^J \omega_i h^2 \|u\|_{H^1(\Omega_i)}^2 + h \|u - P_\Gamma^k u\|_{L^2_\omega(\Gamma)}^2 \\ &\lesssim h^2 |u|_{H^1_\omega(\Omega)}^2 + h \|u - P_\Gamma^k u\|_{L^2_\omega(\Gamma)}^2. \end{aligned}$$

The desired result then follows since

$$\|u - Q_h^\omega u\|_{L^2_\omega(\Omega)} \leq \|u - w\|_{L^2_\omega(\Omega)}.$$

□

From the above proof, we see that the validity of Lemma 4.2 has nothing to do with the cross points. Nevertheless we only know its application to the case that the interface has no internal cross points.

with the induced norms denoted by $\|\cdot\|_{L^2_\omega(\Omega)}$ and $|\cdot|_{H^1_\omega(\Omega)}$, respectively. Moreover we define a full weighted H^1 norm by

$$\|\cdot\|_{H^1_\omega(\Omega)}^2 = \|\cdot\|_{L^2_\omega(\Omega)}^2 + |\cdot|_{H^1_\omega(\Omega)}^2.$$

We assume that Ω is triangulated by a family of quasiuniform meshes $\{\mathcal{T}_h, h < 1\}$ as described earlier. An additional assumption we make here is that these triangulations will be lined up with the subdomains Ω_i 's. Namely the restriction of each \mathcal{T}_h on each Ω_i is also a triangulation of Ω_i itself.

The weighted L^2 projection $Q_h^\omega : L^2(\Omega) \mapsto S_h$ is defined by

$$(4.4) \quad (Q_h^\omega u, v)_{L^2_\omega(\Omega)} = (u, v)_{L^2_\omega(\Omega)}, \quad \forall u \in L^2(\Omega), v \in S_h.$$

We will derive error estimates for Q_h^ω of the following type:

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \leq Ch |\log h|^\gamma |u|_{H^1_\omega(\Omega)}, \quad \forall u \in H^1_0(\Omega)$$

for some positive constant γ . The point here is that we require that the constant C appearing in the above estimate does not depend on the weights $\{\omega_i\}$. Again we will use the notation “ \lesssim ” in place of “ $\leq C$ ”, where C is in particular independent of the weights.

The derivation of such an estimate is not as simple as it might appear. For example, the argument used in the proof of Theorem 3.2 cannot be applied easily here, even though we can get the estimates analogous to (3.4) and (3.5) with proper weights. It is unclear if the interpolation between weighted H^2 and L^2 spaces would give rise to the right space. Nevertheless the proof for $d = 1$ is almost trivial. To be more precise, we have the following

PROPOSITION 4.1. *For $d = 1$, we have for all $u \in H^1_0(\Omega)$*

$$\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \lesssim h |u|_{H^1_\omega(\Omega)},$$

and

$$|Q_h^\omega u|_{H^1_\omega(\Omega)} \lesssim |u|_{H^1_\omega(\Omega)}.$$

Proof. Since $d = 1$, we have $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$. Hence the nodal value interpolant $I_h : C(\bar{\Omega}) \mapsto S_h$ is well-defined in H^1 . It is well-known that (cf. [5]), for any $\tau \in \mathcal{T}_h$

$$\|u - I_h u\|_{L^2(\tau)}^2 \lesssim h^2 |u|_{H^1(\tau)}^2, \quad \forall u \in H^1(\tau).$$

Summing up over all $\tau \in \mathcal{T}_h$ with proper weights, we then get

$$\|(I - I_h)u\|_{L^2_\omega(\Omega)} \leq Ch |u|_{H^1_\omega(\Omega)}, \quad \forall u \in H^1_0(\Omega).$$

Our first inequality then follows since $\|(I - Q_h^\omega)u\|_{L^2_\omega(\Omega)} \leq \|(I - I_h)u\|_{L^2_\omega(\Omega)}$. The proof of the second inequality is similar to Theorem 3.4 by Lemma 3.3. \square

The above approach can not in general be extended to higher dimensions because of the lack of the imbedding $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$, although a similar technique can be applied, as is done in Section 4.2.1 below in some special circumstances when $d = 2$. The analysis for more general cases, especially for $d = 3$, is more complicated and special techniques are needed.

Proof. It follows from the inverse inequality, Lemma 3.2, and Lemma 3.3, that

$$\begin{aligned}
|Q_h u|_{H^1(\Omega)}^2 &= \sum_{\tau \in \mathcal{T}_h} |Q_h u|_{H^1(\tau)}^2 \\
&\leq 2 \sum_{\tau \in \mathcal{T}_h} \{|Q_h u - Q_\tau u|_{H^1(\tau)}^2 + |Q_\tau u|_{H^1(\tau)}^2\} \\
&\lesssim \sum_{\tau \in \mathcal{T}_h} \{h^{-2} |Q_h u - Q_\tau u|_{L^2(\tau)}^2 + |u|_{H^1(\tau)}^2\} \\
&\lesssim \sum_{\tau \in \mathcal{T}_h} \{h^{-2} \|u - Q_\tau u\|_{L^2(\tau)}^2 + |u|_{H^1(\tau)}^2\} + h^{-2} \|u - Q_h u\|_{L^2(\Omega)}^2 \\
&\lesssim |u|_{H^1(\Omega)}^2.
\end{aligned}$$

The desired result then follows. \square

(3.12) Remark. 3.1 Notice that our proof of (3.11) which uses Q_τ is carried out element by element. Such a “local” argument is crucial for us to establish the corresponding stability for the weighted L^2 projection (with trivial modification).

Simultaneous Approximation Properties. From the estimates we derived for Q_h , property (3.3) then becomes clear by Lemma 3.1. In fact such simultaneous approximation property holds for more general boundary conditions. For example, if $\Gamma_0 \subset \partial\Omega$ is measurable, then we have

PROPOSITION 3.5. *For any $u \in H_{\Gamma_0}^1(\Omega)$, there exists $v_h \in S_h \cap H_{\Gamma_0}^1(\Omega)$ such that (3.3) holds.*

4. Weighted L^2 Projection. This section, which is the core of the paper, is devoted to the analysis of the weighted L^2 projections. Both the L^2 error estimates and H^1 stability will be investigated.

Assume the domain Ω admits the following decomposition:

$$(4.1) \quad \bar{\Omega} = \bigcup_{i=1}^J \bar{\Omega}_i,$$

where Ω_i are mutually disjoint. Let Γ denote the set of interfaces, namely

$$\Gamma = \cup_{i=1}^J \partial\Omega_i \setminus \partial\Omega.$$

For simplicity, we assume that Γ consists only of segments ($d = 2$) or plane polygons ($d = 3$). In other words, no part of any $\partial\Omega_i$ is curved.

Given a set of positive constants $\{\omega_i\}_{i=1}^J$, we introduce the following weighted inner products:

$$(4.2) \quad (u, v)_{L_\omega^2(\Omega)} = \sum_{i=1}^J \omega_i (u, v)_{L^2(\Omega_i)},$$

and

$$(4.3) \quad (u, v)_{H_\omega^1(\Omega)} = \sum_{i=1}^J \omega_i \int_{\Omega_i} \nabla u \cdot \nabla v dx,$$

If $Q_{\hat{\tau}}$ is defined similarly, it is then straightforward to verify that

$$(3.7) \quad \widehat{Q_{\tau}u} = Q_{\hat{\tau}}\hat{u}.$$

These locally defined operators have the desired stability and approximation properties as shown by

LEMMA 3.3. *For any $\tau \in \mathcal{T}_h$*

$$(3.8) \quad |Q_{\tau}u|_{H^1(\tau)} \lesssim |u|_{H^1(\tau)}, \quad \forall u \in H^1(\tau),$$

and

$$(3.9) \quad \|u - Q_{\tau}u\|_{L^2(\tau)} \lesssim h|u|_{H^1(\tau)}, \quad \forall u \in H^1(\tau).$$

Proof. It follows from (3.7) that (3.8) is equivalent to

$$(3.10) \quad |Q_{\hat{\tau}}\hat{u}|_{H^1(\hat{\tau})} \lesssim |\hat{u}|_{H^1(\hat{\tau})}, \quad \forall \hat{u} \in H^1(\hat{\tau}).$$

As all the norms on $\mathcal{P}_1(\hat{\tau})$ are equivalent, we have

$$|Q_{\hat{\tau}}\hat{u}|_{H^1(\hat{\tau})} \lesssim \|Q_{\hat{\tau}}\hat{u}\|_{L^2(\hat{\tau})} \leq \|\hat{u}\|_{L^2(\hat{\tau})} \lesssim \|\hat{u}\|_{H^1(\hat{\tau})},$$

which, since $Q_{\hat{\tau}}\hat{c} = \hat{c}$ for any $\hat{c} \in \mathbb{R}^1$, implies that

$$|Q_{\hat{\tau}}\hat{u}|_{H^1(\hat{\tau})} \lesssim \inf_{\hat{c} \in \mathbb{R}^1} \|\hat{u} + \hat{c}\|_{H^1(\hat{\tau})} \lesssim |\hat{u}|_{H^1(\hat{\tau})}.$$

This proves (3.10) and hence (3.8).

Now we turn to the proof of (3.9). By changing variables and using (3.7), we get

$$\begin{aligned} \|u - Q_{\tau}u\|_{L^2(\tau)} &\lesssim h^{d/2} \|\hat{u} - \hat{Q}_{\hat{\tau}}\hat{u}\|_{L^2(\hat{\tau})} \\ &\lesssim h^{d/2} \inf_{\hat{c} \in \mathbb{R}^1} \|\hat{u} + \hat{c}\|_{H^1(\hat{\tau})} \lesssim h^{d/2} |\hat{u}|_{H^1(\hat{\tau})} \\ &\lesssim h^{d/2} h^{1-d/2} |u|_{H^1(\tau)} \lesssim h|u|_{H^1(\tau)}. \end{aligned}$$

This completes the proof. \square

We are now in a position to state and prove our stability theorem.

THEOREM 3.4. *For all $u \in H_0^1(\Omega)$*

$$(3.11) \quad |Q_h u|_{H^1(\Omega)} \lesssim |u|_{H^1(\Omega)}.$$

More specifically, we have

LEMMA 3.1. (3.1) and (3.2) both hold if and only if (3.3) is true.

The proof of the above result which uses triangle inequality and also the inverse property is straightforward.

The inequality (3.3) has been assumed in some papers on finite elements but it seems that little attention is paid to its proof. The stability of the L^2 projection in the H^1 norm was perhaps first established by Bank and Dupont in [2]. Their proof however requires the full elliptic regularity condition (which is unnecessary). A discussion of this problem in two dimensions may also be found in Crouzeix and Thomée [6]. Recently Scott and Zhang [8] have constructed a kind of interpolation operator for nonsmooth functions that can also be used to give a proof of this result. As we pointed out earlier, it can be directly obtained assuming the simultaneous approximation property (3.3) which is actually the approach that Mandel, McCormick and Bank take in [7]. For avoiding a logical circle, the question remains as to how the simultaneous approximation property is justified. Our approach here is to establish the stability by a different argument and obtain the simultaneous approximation property as a consequence.

L^2 Error Estimates. As we have assumed that $d \leq 3$, the Sobolev imbedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ holds. Therefore the usual nodal value interpolant $I_h : C(\bar{\Omega}) \mapsto S_h$ is well-defined in H^2 . It is well-known that (cf. [5])

$$(3.4) \quad \|u - Q_h u\|_{L^2(\Omega)} \leq \|u - I_h u\|_{L^2(\Omega)} \lesssim h^2 |u|_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega).$$

On the other hand

$$(3.5) \quad \|u - Q_h u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}, \quad \forall u \in L^2(\Omega).$$

An application of the standard interpolation technique to above two estimates yields

THEOREM 3.2. For $u \in H_0^1(\Omega)$

$$(3.6) \quad \|u - Q_h u\|_{L^2(\Omega)} \lesssim h |u|_{H^1(\Omega)}.$$

H^1 Stability. The main ingredient in our analysis is a local L^2 projection $Q_\tau : L^2(\tau) \mapsto \mathcal{P}_1(\tau)$, for any given $\tau \in \mathcal{T}_h$, defined by

$$(Q_\tau u, \phi)_{L^2(\tau)} = (u, \phi)_{L^2(\tau)}, \quad \forall u \in L^2(\tau), \phi \in \mathcal{P}_1(\tau).$$

Let $\hat{\tau}$ be the standard reference element, for any $\tau \in \mathcal{T}_h$, we have an affine diffeomorphism:

$$F_\tau : \hat{\tau} \mapsto \tau.$$

For any function $v \in L^2(\tau)$, we adopt the following standard notation:

$$\hat{v}(\hat{x}) = v(F(\hat{x})), \quad \hat{x} \in \hat{\tau}.$$

□

The most interesting part in Lemma 2.3 is perhaps for $d = 2$. That is just the limiting case in which the Sobolev imbedding fails. The following is another such example.

LEMMA 2.4. *Assume Ω is a polyhedral in \mathbb{R}^3 , then*

$$\|v\|_{L^2(\Gamma)} \lesssim |\log h|^{\frac{1}{2}} \|v\|_{H^1(\Omega)} \quad \forall v \in S^h(\Omega)$$

where Γ is any edge of Ω .

Proof. By breaking the domain Ω into (possibly overlapping) subdomains that have parallel faces in one direction, we may assume here, without loss of generality, that $\Omega = (0,1)^3$ is the unit cube. Applying the inequality in Lemma 2.2 with the domain $D = (0,1)^2$ and $w = v(x_1, x_2, x_3)$, we get

$$|v(0,0,x_3)|^2 \lesssim |\log \epsilon| \int_D (v^2 + \sum_{i=1}^2 |\frac{\partial v}{\partial x_i}|^2) dx_1 dx_2 + \epsilon^2 \|v\|_{W^{1,\infty}(\Omega)}^2, \quad \forall \epsilon \in (0,1).$$

Integrating with respect to x_3 , we get

$$\int_0^1 |v(0,0,x_3)|^2 dx_3 \lesssim |\log \epsilon| \|v\|_{H^1(\Omega)}^2 + \epsilon^2 \|v\|_{W^{1,\infty}(\Omega)}^2, \quad \forall \epsilon \in (0,1).$$

Taking $\epsilon = h^{\frac{3}{2}}$ and applying the inverse inequality yield

$$\|v\|_{L^2(\Gamma)} \lesssim |\log h|^{\frac{1}{2}} \|v\|_{H^1(\Omega)} \quad \forall v \in S^h(\Omega)$$

where $\Gamma = \{(0,0,x_3) : 0 \leq x_3 \leq 1\}$.

Similar arguments obviously apply to the other edges of Ω , the proof is then complete. □

3. Ordinary L^2 Projections. In this section, we shall consider the usual L^2 projection with respect to the ordinary L^2 inner product (namely without weights).

Associated with the finite element space S_h , the L^2 projection $Q_h : L^2(\Omega) \mapsto S_h$ is defined by

$$(Q_h u, v) = (u, v), \quad \forall u \in L^2(\Omega), v \in S_h.$$

The aim of this section is to establish some estimates for Q_h on H^1 in both the L^2 and H^1 norms, namely for all $u \in H_0^1(\Omega)$

$$(3.1) \quad \|u - Q_h u\|_{L^2(\Omega)} \lesssim h |u|_{H^1(\Omega)},$$

and

$$(3.2) \quad |Q_h u|_{H^1(\Omega)} \lesssim |u|_{H^1(\Omega)}.$$

The above estimates are closely related to the so-called simultaneous approximation property:

$$(3.3) \quad \inf_{\chi \in S_h} (\|u - \chi\|_{L^2(\Omega)} + h \|u - \chi\|_{H^1(\Omega)}) \lesssim h \|u\|_{H^1(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

LEMMA 2.2. *Assume D is bounded domain in \mathbb{R}^2 with ∂D Lipschitz continuous. Then*

$$\|w\|_{L^\infty(D)} \lesssim |\log \epsilon|^{\frac{1}{2}} \|w\|_{H^1(D)} + \epsilon \|w\|_{W^{1,\infty}(D)}, \quad \forall w \in W^{1,\infty}(D), \epsilon \in (0, 1).$$

Next we introduce the finite element space. For $0 < h < 1$, let \mathcal{T}_h be a triangulation of $\bar{\Omega}$ with simplices K of diameter less than or equal to h . We assume the family $\{\mathcal{T}_h\}$ is quasiuniform, namely there are constants $c_0 > 0$ and $c_1 > 0$ such that

$$\max_{K \in \mathcal{T}_h} \frac{h_K}{\rho_K} \leq c_0, \quad \frac{\max_{K \in \mathcal{T}_h} h_K}{\min_{K \in \mathcal{T}_h} h_K} \leq c_1 \quad \forall h,$$

where h_K is the diameter of K and ρ_K is the diameter of the largest ball contained in K . Corresponding to each triangulation \mathcal{T}_h , we define a finite element subspace $S_h \subset H_0^1(\Omega)$ that consists of continuous piecewise (with respect to the elements in \mathcal{T}_h) linear polynomials vanishing on $\partial\Omega$.

For a given triangulation \mathcal{T}_h , we consider a finer quasiuniform mesh $\underline{\mathcal{T}}_h$ with $\underline{h} < h$ which is obtained by refining \mathcal{T}_h in such a way that

$$S_h \subset S_{\underline{h}}$$

where $S_{\underline{h}} \subset H_0^1(\Omega)$ is the corresponding finite element space defined on $\underline{\mathcal{T}}_h$.

It is well-known that, for any function $v \in S_h$

$$(2.2) \quad \|v\|_{L^2(\Omega)} \asymp h^d \sum_{x \in \mathcal{N}_h} v^2(x),$$

where \mathcal{N}_h is the set of vertices of the triangulation \mathcal{T}_h , and

$$(2.3) \quad \|v\|_{L^\infty(\Omega)} \asymp h^{-d/p} \|v\|_{L^p(\Omega)} (1 \leq p \leq \infty).$$

The right hand side of (2.2) is often called the discrete L^2 norm. Inequality (2.3) is the well-known inverse property of the finite element spaces (cf. [5]).

LEMMA 2.3. *For all $v \in S_h(\Omega)$,*

$$(2.4) \quad \|v\|_{L^\infty(\Omega)} \lesssim \begin{cases} \|v\|_{H^1(\Omega)}, & \text{if } d = 1; \\ |\log h|^{\frac{1}{2}} \|v\|_{H^1(\Omega)}, & \text{if } d = 2; \\ h^{-\frac{1}{2}} \|v\|_{H^1(\Omega)}, & \text{if } d = 3. \end{cases}$$

Proof. The first inequality (for $d = 1$) follows from the usual Sobolev imbedding (in (2.1)). The second one is well-known in the literature (cf. [3]) and it can be easily obtained by using Lemma 2.2 together with the inverse inequality (2.3) with $\epsilon = h$.

The proof of the last case for $d = 3$ is also almost trivial and can be furnished by using the inverse inequality (2.3) and Sobolev imbedding (2.1):

$$\|v\|_{L^\infty(\Omega)} \lesssim h^{-\frac{1}{2}} \|v\|_{L^6(\Omega)} \lesssim h^{-\frac{1}{2}} \|v\|_{H^1(\Omega)}.$$

The remainder of the paper is organized as follows. In Section 2, some preliminary materials, such as Sobolev space, finite element space etc., will be presented. Section 3 is devoted to the analysis of the usual L^2 projection. Main estimates for weighted L^2 projections will be presented in Section 4.

2. Preliminaries. Let $\Omega \subset \mathbb{R}^d$ ($1 \leq d \leq 3$) be a bounded domain. For simplicity, we assume that Ω is an interval for $d = 1$, a polygon for $d = 2$ and a polyhedron for $d = 3$. On Ω , $L^p(\Omega)$ denotes the usual Banach space consisting of p -th power integrable functions. The Sobolev space of index (m, p) is defined by

$$W^{m,p}(\Omega) \stackrel{def}{=} \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \text{ if } |\alpha| \leq m\},$$

with a norm

$$\|v\|_{W^{m,p}(\Omega)} \stackrel{def}{=} \left(\sum_{|\alpha| \leq m} \|D^\alpha v\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}},$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ is multi-integer and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}, \quad |\alpha| = \sum_{i=1}^d \alpha_i.$$

For $p = 2$, by convention, we denote

$$H^m(\Omega) \stackrel{def}{=} W^{m,2}(\Omega).$$

We will have occasion to use the following seminorms:

$$|v|_{W^{m,p}(\Omega)} \stackrel{def}{=} \left(\sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}.$$

For $m = 1$, $H_0^1(\Omega)$ denotes the subspace of $H^1(\Omega)$ consisting of functions that vanish on $\partial\Omega$ in an appropriate sense. Similarly, for a measurable $\Gamma_0 \subset \partial\Omega$, $H_{\Gamma_0}^1(\Omega)$ is the space consisting of functions in H^1 that vanish on Γ_0 .

We quote the following well-known Sobolev continuous imbeddings ([1]):

$$(2.1) \quad H^1(\Omega) \hookrightarrow \begin{cases} L^\infty(\Omega), & \text{if } d = 1; \\ L^p(\Omega) (1 \leq p < \infty), & \text{if } d = 2; \\ L^6(\Omega), & \text{if } d = 3. \end{cases}$$

LEMMA 2.1.

$$\|u\|_{L^2(\partial\Omega)} \lesssim \epsilon^{-1} \|u\|_{L^2(\Omega)} + \epsilon \|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega) \text{ and } \epsilon \in (0, 1).$$

For a proof of the above result, we refer to [10].

The following result is a special case of Theorem 2.1 in [10]. (cf. also [9].)

SOME ESTIMATES FOR A WEIGHTED L^2 PROJECTION *

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Abstract. This paper is devoted to the error estimates for some weighted L^2 projections. Nearly optimal estimates are obtained. These estimates can be applied to the analysis of the usual multigrid method, multilevel preconditioner and domain decomposition method for solving elliptic boundary problem whose coefficients have large jump discontinuities.

Key Words. Finite element space, weighted L^2 projections, multigrid, domain decomposition

AMS(MOS) subject classification. 65M60, 65N15, 65N30

1. Introduction. This work was motivated by the study of the numerical solution of elliptic boundary value problems that has large discontinuity jumps in coefficients. If these jumps become larger, the corresponding discretized (by finite element for example) equations may be harder to solve. In some special cases, however, multigrid or domain decomposition method can be properly designed so that the numerically observed convergence rate is actually independent of these jumps. We find that the theoretical justification of this phenomenon lies in certain approximation and stability properties of some weighted L^2 projections with weights provided by the discontinuous coefficients (cf. [10], [11] and [4]). The point is that we want to get estimates which are uniform with respect to the weights. It is quite easy or difficult, depending how you look at it, to be convinced that this kind of estimate is nontrivial based on known theory in the finite element method.

A careful study of this type of weighted L^2 projection will be made in this paper. We shall establish estimates that are nearly optimal under some special circumstances. In a sequel of this paper, we shall present some negative results to demonstrate that the expected estimates are not always possible in general and the results in the paper are sharp in certain sense.

Related to the topic of this paper is the usual L^2 projection. Some error and stability estimates for such a projection are also presented with complete proofs.

As is done in [10], we will use the following notation:

$$x \lesssim y, \quad f \gtrsim g \quad \text{and} \quad u \asymp v$$

which means that

$$x \leq Cy \quad , \quad f \geq cg \quad \text{and} \quad cv \leq u \leq Cv$$

where C and c , are positive constants independent of the variables appearing in the inequalities and any other parameters related to meshes, spaces etc.

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