Preliminary version: September 8, 1999

On the stability of the L_2 projection in $H^1(\Omega)$*

J. H. Bramble, J. E. Pasciak

Department of Mathematics, Texas A&M University,
College Station, TX, 77843, U.S.A.
{bramble,pasciak}@math.tamu.edu

O. Steinbach

Mathematisches Institut A, Universität Stuttgart,
Pfaffenwaldring 57, 70569 Stuttgart, Germany
steinbach@mathematik.uni-stuttgart.de

Abstract

We prove the stability of the L_2 projection onto the finite element space of piecewise linear basis functions in $H^1(\Omega)$ assuming appropriate mesh conditions locally. We give explicit formulae to check these conditions locally for a given finite element mesh.

Key words: L_2 projection, finite elements, stability, adaptivity.
AMS subject classifications: 65D05, 65N30, 65N50.

1 Introduction

Let $V_h = \text{span}\{\varphi_k\}_{k=1}^M \subset H^1(\Omega)$ be the finite element trial space of piecewise linear continuous basis functions where $\Omega \subset \mathbb{R}^n$ $(n = 1, 2, 3)$ is a bounded domain. For simplicity we assume that $\partial\Omega$ is a polygon $(n = 2)$ or a polyhedron $(n = 3)$. The L_2 projection Q_h of a given function u onto the finite element space V_h is defined by

$$\langle Q_h u, v^h \rangle_{L_2(\Omega)} = \langle u, v^h \rangle_{L_2(\Omega)} \quad \text{for all } v^h \in V_h. \tag{1.1}$$

*This work was done while the third author was a Postdoctoral Research Associate at the Institute for Scientific Computation (ISC), Texas A & M University, College Station. The financial support by the ISC is gratefully acknowledged.

From (1.1) it is obvious that the projection $Q_h : L_2(\Omega) \to V_h \subset L_2(\Omega)$ is bounded, i.e. $||Q_h u||_{L_2(\Omega)} \le ||u||_{L_2(\Omega)}$ holds for all $u \in L_2(\Omega)$ without any further condition on the trial space V_h. In this paper we are concerned with the stability of the L_2 projection as a map $Q_h : H^1(\Omega) \to V_h \subset H^1(\Omega)$, in particular we will prove the stability estimate

$$||Q_h u||_{H^1(\Omega)} \le c \cdot ||u||_{H^1(\Omega)} \qquad \text{for all } u \in H^1(\Omega) \tag{1.2}$$

assuming some appropriate conditions on the given finite element trial space V_h, in paticular, on the underlying triangulation.

The stability estimate (1.2) of the L_2 projection Q_h is of general interest in the numerical analysis, in particular for Galerkin finite or boundary element methods for elliptic and parabolic boundary value problems [3, 5, 12]. For example, the stability estimate (1.2) is needed to analyse the numerical realisation of a Neumann series according to a second kind boundary integral equation, see [8].
Using interpolation arguments we get

$$||Q_h u||_{H^s(\Omega)} \le c \cdot ||u||_{H^s(\Omega)} \qquad \text{for all } u \in H^s(\Omega), \quad s \in [0,1]. \tag{1.3}$$

From (1.3) we can conclude the stability estimate, see for example [6],

$$c \cdot ||u^h||_{H^s(\Omega)} \le \sup_{v^h \in V_h} \frac{|\langle u^h, v^h \rangle_{L_2(\Omega)}|}{||v^h||_{\tilde{H}^{-s}(\Omega)}} \quad \text{for all } u^h \in V_h \tag{1.4}$$

where $\tilde{H}^{-s}(\Omega) = (H^s(\Omega))'$ is defined by duality with respect to the L_2 inner product. The estimate (1.4) is esentially needed in the design of hybrid boundary element methods [10] as well as in the construction of efficient preconditioners in finite and boundary element methods [11].

For global quasi–uniform triangulations Ω_h, where a global inverse inequality is valid, the estimate (1.2) is a direct consequence of the latter. In [5] the stability estimate (1.2) was shown for non–uniform triangulations in one and two dimensions satisfying certain mesh conditions. While for $n = 1$ explicit restrictions on the mesh partition are formulated, such conditions are not so obvious for $n = 2$.

In this paper we prove the stability estimate (1.2) for arbitrary $n = 1, 2, 3$ based on local stability conditions to be satisfied. This approach is valid for general trial spaces, in particular, trial functions of arbitrary polynomial order, however, here we consider the case of piecewise linear basis functions only. In this situation we are able to formulate local mesh assumptions which ensure the stability estimate (1.2). For a given finite element mesh Ω_h the local mesh conditions can be computed in an explicit form, in particular, we are able to improve the mesh by local regularizations if needed. This can be done either by moving mesh nodes or by some additional refinement locally.

The remainder of this paper is organized as follows. Some preliminary notations are given in Section 2. In Section 3 we recall from [4, 7] the definition as well as some error and stability estimates for a quasi interpolation operator needed in our analysis. The main result is formulated in Theorem 4.1. The proof is based on a general stability condition and several technical results given in Section 5. In Section 6 we discuss the stability condition in case of piecewise linear basis functions. Based on the eigenvalue analysis of weighted Gram matrices defined locally we derive computable criteria when the stability condition is satisfied.

2 Notations

Let

$$\Omega_h = \bigcup_{\ell=1}^{N} \tau_\ell, \qquad \Delta_\ell := \int_{\tau_\ell} dx \quad \text{for } \ell = 1, \ldots, N \tag{2.1}$$

be a family of locally quasi–uniform finite element meshes, i.e., a finite element τ_h is either an interval ($n = 1$), a triangle ($n = 2$) or a tetrahedron ($n = 3$). Assuming for $n > 1$ that all angles inside an element τ_ℓ are bounded below independent of N we can define a local mesh size to be

$$h_\ell := \Delta_\ell^{1/n} \qquad \text{for } \ell = 1, \ldots, N. \tag{2.2}$$

Let $\{x_k\}$ the set of all nodes of the mesh Ω_h. For any piecewise linear basis function $\varphi_k \subset V_h$, which is assumed to be related to a node x_k, we define

$$\omega_k := \operatorname{supp} \varphi_k, \quad k = 1, \ldots, M. \tag{2.3}$$

Let $I(k)$ denote the index set of all elements τ_ℓ satisfying $\tau_\ell \subset \omega_k$. Then we define a local mesh size asscociated to the basis function φ_k as

$$\hat{h}_k := \frac{1}{\# I(k)} \sum_{\ell \in I(k)} h_\ell \quad \text{for } k = 1, \ldots, M. \tag{2.4}$$

Since the mesh Ω_h is assumed to be locally quasi–uniform, there exists a positive constant $c \geq 1$ independent of k, M and N such that

$$\frac{1}{c} \leq \frac{\hat{h}_k}{h_\ell} \leq c \quad \text{for all } \ell \in I(k), \quad k = 1, \ldots, M. \tag{2.5}$$

We define $J(\ell)$ to be the index set of all basis functions $\varphi_k \in V_h$ where $x_k \in \tau_\ell$ is satisfied. Note that there holds an inverse inequality locally [3], in particular

$$||v^h||_{H^1(\tau_\ell)} \leq c \cdot h_\ell^{-1} \cdot ||v^h||_{L_2(\tau_\ell)} \quad \text{for all } v^h \in V_h, \ \ell = 1, \ldots, N, \tag{2.6}$$

with a constant c independent of v^h and h_ℓ.

3 Quasi interpolation

To prove the stability estimate (1.2) we need to use a projection operator P_h which is stable in $H^1(\Omega)$ and which provides local error estimates in $L_2(\tau_\ell)$ valid on all finite elements τ_ℓ for $\ell = 1, \ldots, N$. For this we will use the concept of quasi interpolation operators first introduced by Clement in [4], see also [7].
We define local trial spaces of piecewise linear continuous trial functions as

$$V_k := \text{span}\{\varphi_j^{(k)}\}_{j=1}^{M_k} \subset H^1(\omega_k), \tag{3.1}$$

in particular, $V_k = V_{h|\omega_k}$ is the restriction of V_h onto ω_k. The local L_2 projections onto V_k are defined by

$$\langle P_k u, v\rangle_{L_2(\omega_k)} = \langle u, v\rangle_{L_2(\omega_k)} \quad \text{for all } v \in V_k\,. \tag{3.2}$$

From (3.2) there follows directly the stability estimate

$$||P_k u||_{L_2(\omega_k)} \leq ||u||_{L_2(\omega_k)} \tag{3.3}$$

as well as the local error estimate

$$||(I - P_k)u||_{L_2(\omega_k)} \leq c \cdot \hat{h}_k \cdot |u|_{H^1(\omega_k)}. \tag{3.4}$$

Moreover, since the mesh is assumed to be locally quasiuniform, we have

$$||P_k u||_{H^1(\omega_k)} \leq c \cdot ||u||_{H^1(\omega_k)} \quad \text{for all } u \in H^1(\omega_k), \ k = 1, \ldots, M, \tag{3.5}$$

where the constant c is independent of $\hat{h}_k$ and k, see, e.g., [2].
Now we define a quasi interpolation operator as

$$(P_h u)(x) \,=\, \sum_{k=1}^{M} (P_k u)(x_k) \cdot \varphi_k(x)\,. \tag{3.6}$$

It is easy to check that P_h is a projection. Moreover, P_h is stable in $H^1(\Omega)$ and provides some local error estimates:

Lemma 3.1 *Let u be in $H^1(\Omega)$. There exists a positive constant c independent of $\ell = 1, \ldots, N$ such that*

$$||(I - P_h)u||_{L_2(\tau_\ell)} \,\le\, c \cdot \sum_{k \in J(\ell)} \hat{h}_k \cdot ||u||_{H^1(\omega_k)} \quad \textit{for all } \ell = 1, \ldots, N. \tag{3.7}$$

Moreover,

$$||P_h u||_{H^1(\Omega)} \,\le\, c \cdot ||u||_{H^1(\Omega)} \quad \textit{for all } u \in H^1(\Omega). \tag{3.8}$$

Proof. The proof follows the general ideas already given in [4]. However, we will give the proof for the specific quasi interpolation operator P_h as defined in (3.6). Let τ_ℓ be an arbitrary but fixed finite element and let $\tilde{k} \in J(\ell)$ be a fixed index. For $x \in \tau_\ell$ we have the representation

$$(P_h u)(x) \,=\, (P_{\tilde{k}} u)(x) + \sum_{\tilde{k} \neq k \in J(\ell)} \left[(P_k u)(x_k) - (P_{\tilde{k}} u)(x_k) \right] \varphi_k(x).$$

Let $s = 0, 1$. Note that

$$||\varphi_k||_{H^s(\tau_\ell)} \,\le\, c \cdot h_\ell^{n/2-s}\,.$$

Then, using the stability estimate (3.5) and the local error estimate (3.4), it follows that

$$\begin{aligned} ||(I - P_h)u||_{H^s(\tau_\ell)} \;&\le\; c_1 \cdot \hat{h}_{\tilde{k}}^{1-s} \cdot ||u||_{H^1(\omega_{\tilde{k}})} \\ &\quad + c_2 \cdot h_\ell^{n/2-s} \sum_{\tilde{k} \neq k \in J(\ell)} |(P_k u)(x_k) - (P_{\tilde{k}} u)(x_k)| \end{aligned}$$

Now

$$||v^h||_{L_\infty(\tau_\ell)} \,\le\, c \cdot h_\ell^{-n/2} \cdot ||v^h||_{L_2(\tau_\ell)} \quad \text{for all } v^h \in V_h, \quad \ell = 1, \ldots, N$$

and (3.4) we get for $x_k \in \tau_\ell$,

$$
\begin{aligned}
|(P_k u)(x_k) - (P_{\tilde{k}} u)(x_k)| \quad &\leq \quad ||P_k u - P_{\tilde{k}} u||_{L_\infty(\tau_\ell)} \\
&\leq c \cdot h_\ell^{-n/2} \cdot ||P_k u - P_{\tilde{k}} u||_{L_2(\tau_\ell)} \\
&\leq c \cdot h_\ell^{-n/2} \cdot \left\{ ||(I - P_k)u||_{L_2(\tau_\ell)} + ||(I - P_{\tilde{k}})u||_{L_2(\tau_\ell)} \right\} \\
&\leq c \cdot h_\ell^{-n/2} \cdot \left\{ \hat{h}_k \cdot ||u||_{H^1(\omega_k)} + \hat{h}_{\tilde{k}} \cdot ||u||_{H^1(\omega_{\tilde{k}})} \right\}.
\end{aligned}
$$

Hence,

$$
||(I - P_h)u||_{H^s(\tau_\ell)} \;\leq\; c \cdot \sum_{k \in J(\ell)} \hat{h}_k^{1-s} \cdot ||u||_{H^1(\omega_k)}
$$

for $s = 0, 1$ and $\ell = 1, \ldots, N$. Using this estimate for $s = 0$ gives the local estimate (3.7) while for $s = 1$ we get the stability estimate (3.8) when assembling over all finite elements τ_ℓ. ■

4 Main results

In this section we will formulate and prove the main result of this paper, in particular, the stability estimate (1.2) assuming some appropriate mesh assumptions. For this we define local weights as

$$
\gamma_k \;:=\; \sqrt{\sum_{\ell \in I(k)} h_\ell^{-2} \cdot ||\varphi_k||^2_{L_2(\tau_\ell)}} \quad \text{for } k = 1, \ldots, M \tag{4.1}
$$

as well as some diagonal matrices of dimension M given by

$$
D_\gamma \;=\; \operatorname{diag}(\gamma_k), \quad D_\varphi \;=\; \operatorname{diag}\left(\hat{h}_k \cdot ||\varphi_k||_{L_2(\Omega)}\right), \quad H \;=\; \operatorname{diag}\left(\hat{h}_k\right) \tag{4.2}
$$

where $\hat{h}_k$ is defined as in (2.4).
Moreover, for each element τ_ℓ we define local matrices G_ℓ, D_ℓ and H_ℓ given as

$$
\begin{aligned}
G_\ell[j,i] \quad &= \quad \langle \varphi_i, \varphi_j \rangle_{L_2(\tau_\ell)} \quad \text{for } i, j \in J(\ell), \\
D_\ell \quad &= \quad \operatorname{diag}\left(||\varphi_i||^2_{L_2(\tau_\ell)}\right)_{i \in J(\ell)}, \\
H_\ell \quad &= \quad \operatorname{diag}\left(\hat{h}_i\right)_{i \in J(\ell)}.
\end{aligned}
$$

Now we are able to formulate local assumptions to be used in the remainder of this section: In what follows we assume that there holds

$$(H_\ell^{-1} G_\ell H_\ell \underline{x}_\ell, \underline{x}_\ell) \;\geq\; c_0 \cdot (D_\ell \underline{x}_\ell, \underline{x}_\ell) \quad \text{for all } \underline{x}_\ell \in I\!R^{\# J(\ell)}. \tag{4.3}$$

In case of a global uniform mesh we have $H = h \cdot I$ and therefore (4.3) is well known. For global non–uniform finite element meshes assumption (4.3) is a criteria for the non–uniformity allowed locally.
The next result is crucial for the proof of the main result given in Theorem 4.1. A similar estimate was used in [1] to construct spectrally equivalent multilevel preconditioners in finite element methods. While for globally uniform triangulations both results are equivalent, this is not the case for non–uniform meshes.

Lemma 4.1 *Let assumption (4.3) be satisfied and $\varphi_k \in V_h$ for $k = 1, \ldots, M$. Then there exists a positive constant c such that*

$$\sum_{\ell=1}^{N} h_\ell^{-2} \cdot ||v^h||^2_{L_2(\tau_\ell)} \;\leq\; c \cdot \sum_{k=1}^{M} \left[\frac{\langle v^h, \varphi_k \rangle_{L_2(\Omega)}}{\hat{h}_k \cdot ||\varphi_k||_{L_2(\Omega)}} \right]^2 \tag{4.4}$$

for all $v^h \in V_h$.

Now we are in the position to formulate and to prove the main result of this paper, in particular the stability estimate (1.2).

Theorem 4.1 *Let assumption (4.3) be satisfied. Then the L_2 projection $Q_h : H^1(\Omega) \to V_h \subset H^1(\Omega)$ is stable, in particular, there exists a positive constant c independent of v and N such that*

$$||Q_h v||_{H^1(\Omega)} \;\leq\; c \cdot ||v||_{H^1(\Omega)} \quad \textit{for all } v \in H^1(\Omega). \tag{4.5}$$

Proof. Using the triangle inequality, the stability estimate (3.8) and the inverse inequality (2.6) we get

$$\begin{aligned} ||Q_h v||^2_{H^1(\Omega)} &\leq 2 \cdot \left\{ ||P_h v||^2_{H^1(\Omega)} + \sum_{\ell=1}^{N} ||Q_h - P_h||^2_{H^1(\tau_\ell)} \right\} \\ &\leq c \cdot \left\{ ||v||^2_{H^1(\Omega)} + \sum_{\ell=1}^{N} h_\ell^{-2} \cdot ||(Q_h - P_h) v||^2_{L_2(\tau_\ell)} \right\}. \end{aligned}$$

From Lemma 4.1, definition (1.1) of the L_2 projection Q_h and local Schwarz inequalities with respect to ω_k, it follows that

$$\begin{aligned}\sum_{\ell=1}^{N} h_\ell^{-2} \cdot ||(Q_h - P_h)v||^2_{L_2(\tau_\ell)} &\le c \cdot \sum_{k=1}^{M} \left[\frac{\langle (Q_h - P_h)v, \varphi_k \rangle_{L_2(\Omega)}}{\hat{h}_k \cdot ||\varphi_k||_{L_2(\Omega)}} \right]^2 \\ &= c \cdot \sum_{k=1}^{M} \left[\frac{\langle (I - P_h)v, \varphi_k \rangle_{L_2(\omega_k)}}{\hat{h}_k \cdot ||\varphi_k||_{L_2(\Omega)}} \right]^2 \\ &\le c \cdot \sum_{k=1}^{M} \hat{h}_k^{-2} \cdot ||(I - P_h)v||^2_{L_2(\omega_k)}\end{aligned}$$

Hence, the assertion follows from (3.7) and the local quasi–uniformity of the finite element mesh Ω_h. ■

5 Proofs

In this section we prove Lemma 4.1. We show first the invertibility of a scaled Gram matrix needed in the proof of Lemma 4.1.

Lemma 5.1 *Let assumption (4.3) be satisfied. Then there exists a positive constant c such that*

$$||\underline{x}||_2 \le c \cdot ||A\underline{x}||_2 \qquad \text{for all } \underline{x} \in I\!R^M$$

where A is the scaled Gram matrix defined by

$$A = D_\varphi^{-1} G D_\gamma^{-1}. \tag{5.6}$$

Proof. For $\underline{u} \in I\!R^M$ we define $\underline{v} = H\underline{u} \in I\!R^M$ and $\underline{w} = H^{-1}\underline{u} \in I\!R^M$ associated with functions

$$v^h = \sum_{k=1}^{M} \hat{h}_k u_k \varphi_k \in V_h, \quad w^h = \sum_{k=1}^{M} \hat{h}_k^{-1} u_k \varphi_k \in V_h\,.$$

Hence, using Assumption (4.3) and $\widetilde{G} = H^{-1}GH$,

$$\begin{aligned}(\widetilde{G}\underline{u}, \underline{u}) &= (G\underline{v}, \underline{w}) = \langle v^h, w^h \rangle_{L_2(\tau_\ell)} = \sum_{\ell=1}^{N} \langle v^h, w^h \rangle_{L_2(\tau_\ell)} \\ &= \sum_{\ell=1}^{N} (H_\ell^{-1} G_\ell H_\ell \underline{u}_\ell, \underline{u}_\ell) \ge c_0 \cdot \sum_{\ell=1}^{N} (D_\ell \underline{u}_\ell, \underline{u}_\ell) = c_0 \cdot (D\underline{u}, \underline{u}).\end{aligned}$$

Note that the matrix $D^{1/2} = \mathrm{diag}(||\varphi_k||_{L_2(\Omega)})$ is well defined. From

$$\begin{aligned} c_0 \cdot ||D^{1/2}\underline{u}||_2^2 &= c_0 \cdot (D\underline{u}, \underline{u}) \le (\widetilde{G}\underline{u}, \underline{u}) \\ &= (D^{-1/2}\widetilde{G}\underline{u}, D^{1/2}\underline{u}) \le ||D^{-1/2}\widetilde{G}\underline{u}||_2 ||D^{1/2}\underline{u}||_2 \end{aligned}$$

we conclude that

$$c_0 \cdot ||D^{1/2}\underline{u}||_2 \le ||D^{-1/2}\widetilde{G}\underline{u}||_2 \quad \text{for all } \underline{u} \in I\!R^M.$$

With the transformation $\underline{\tilde{u}} = D_\gamma \underline{u}$ this is equivalent to

$$c_0 \cdot ||D^{1/2}D_\gamma^{-1}\underline{\tilde{u}}||_2 \le ||D^{-1/2}D_\varphi D_\varphi^{-1}\widetilde{G}D_\gamma^{-1}\underline{\tilde{u}}||_2 = ||D^{-1/2}D_\varphi \widetilde{A}\underline{\tilde{u}}||_2$$

using $\widetilde{A} = D_\varphi^{-1}\widetilde{G}D_\gamma^{-1}$. From

$$\frac{D^{1/2}[k,k]}{D_\gamma[k,k]} = \frac{||\varphi_k||_{L_2(\Omega)}}{\sqrt{\sum_{\ell \in I(k)} h_\ell^{-2} ||\varphi_k||_{L_2(\tau_\ell)}^2}} \ge c \cdot \hat{h}_k$$

and

$$\frac{D_\varphi[k,k]}{D^{1/2}[k,k]} = \frac{\hat{h}_k \cdot ||\varphi_k||_{L_2(\Omega)}}{||\varphi_k||_{L_2(\Omega)}} = \hat{h}_k$$

for all $k = 1, \ldots, M$ we get that

$$c \cdot ||H\underline{\tilde{u}}||_2 \le ||H\widetilde{A}\underline{\tilde{u}}||_2 \quad \text{for all } \underline{\tilde{u}} \in I\!R^M.$$

With the transformation $\underline{x} = H\underline{\tilde{u}}$ and inserting the definitions of $\widetilde{A}$ and $\widetilde{G}$ this gives

$$c \cdot ||x||_2 \le ||H\widetilde{A}H^{-1}\underline{x}||_2 = ||HD_\varphi^{-1}H^{-1}GHD_\gamma^{-1}H^{-1}\underline{x}||_2 = ||A\underline{x}||_2$$

for all $\underline{x} \in I\!R^M$ using that D_φ, D_γ and H are diagonal. ■

Proof of Lemma 4.1

For $v^h = \sum_{k=1}^M v_k \varphi_k \in V_h$ the left hand side in (4.4) gives

$$\begin{aligned} \sum_{\ell=1}^N h_\ell^{-2} \cdot ||v^h||_{L_2(\tau_\ell)}^2 &\le c \cdot \sum_{\ell=1}^N h_\ell^{-2} \sum_{k \in J(\ell)} v_k^2 \cdot ||\varphi_k||_{L_2(\tau_\ell)}^2 \\ &= c \cdot \sum_{k=1}^M v_k^2 \sum_{\ell \in I(k)} h_\ell^{-2} \cdot ||\varphi_k||_{L_2(\tau_\ell)}^2 \\ &= c \cdot \sum_{k=1}^M v_k^2 \gamma_k^2 = c \cdot \sum_{k=1}^M x_k^2 = c \cdot ||\underline{x}||_2^2 \end{aligned}$$

where $x_k = \gamma_k v_k$. The right hand side in (4.4) is

$$\begin{aligned}\sum_{k=1}^{M}\left[\frac{\langle v^h,\varphi_k\rangle_{L_2(\Omega)}}{\hat{h}_k||\varphi_k||_{L_2(\Omega)}}\right]^2 &= \sum_{k=1}^{M}\left[\sum_{j=1}^{M} v_j\cdot\frac{\langle\varphi_j,\varphi_k\rangle_{L_2(\Omega)}}{\hat{h}_k||\varphi_k||_{L_2(\Omega)}}\right]^2\\ &= \sum_{k=1}^{M}\left[\sum_{j=1}^{M} x_j\cdot\frac{\langle\varphi_j,\varphi_k\rangle_{L_2(\Omega)}}{\gamma_j\hat{h}_k||\varphi_k||_{L_2(\Omega)}}\right]^2\\ &= \sum_{k=1}^{M}[(A\underline{x})_k]^2 = ||A\underline{x}||_2^2\end{aligned}$$

using the matrix definition (5.6). Hence, (4.4) follows from Lemma 5.1. ∎

6 Finite element spaces

The stability estimate in Theorem 4.1 is based on the stability assumption (4.3). If we define the symmetric matrix

$$\widetilde{G}_\ell^S := \frac{1}{2}\left[H_\ell G_\ell H_\ell^{-1} + H_\ell^{-1} G_\ell H_\ell\right] \tag{6.1}$$

then assumption (4.3) is equivalent to

$$(\widetilde{G}_\ell^S \underline{x}_\ell, \underline{x}_\ell) = (H_\ell^{-1} G_\ell H_\ell \underline{x}_\ell, \underline{x}_\ell) \geq c_0\cdot(D_\ell \underline{x}_\ell, \underline{x}_\ell) \quad \text{for all } \underline{x}_\ell \in I\!R^{\#J(\ell)}. \tag{6.2}$$

Let $\tau_\ell \in \Omega_h$ be an arbitrary finite element, Note that $\#J(\ell) = n+1$ when using piecewise linear basis functions. A simple computation shows that

$$D_\ell = c_n\cdot\Delta_\ell\cdot I_{n+1} \quad \text{with } c_n = \begin{cases} 1/3, & \text{for } n=1,\\ 1/6, & \text{for } n=2,\\ 1/10, & \text{for } n=3,\end{cases} \tag{6.3}$$

where I_{n+1} is the unit matrix in $n+1$ dimensions. Moreover,

$$\begin{aligned}\langle\varphi_i,\varphi_i\rangle_{L_2(\tau_\ell)} &= c_n\cdot\Delta_\ell \quad \text{for } i=1,\ldots,n+1,\\ \langle\varphi_i,\varphi_j\rangle_{L_2(\tau_\ell)} &= \frac{1}{2}\cdot c_n\cdot\Delta_\ell \quad \text{for } i,j=1,\ldots,n+1, i\neq j.\end{aligned}$$

Therefore,

$$\widetilde{G}_\ell^S = \frac{1}{4}\cdot c_n\cdot\Delta_\ell\cdot A \tag{6.4}$$

with a matrix A defined by

$$A[j,i] \;=\; \begin{cases} 4 & \text{for } i=j, \\ \dfrac{\hat{h}_i}{\hat{h}_j}+\dfrac{\hat{h}_j}{\hat{h}_i} & \text{for } i \neq j, \end{cases} \qquad i,j=1,\ldots,n+1. \tag{6.5}$$

Hence, to show (6.2) it is sufficient to consider the eigenvalues λ_i of the matrix A. Using symbolic computations, we get

$$\lambda_{1,2} \;=\; 3+n \pm \sqrt{\sum_{i=1}^{n+1}\frac{1}{\hat{h}_i^2}\cdot\sum_{i=1}^{n+1}\hat{h}_i^2}, \quad \lambda_j = 2 \quad \text{for } j=3,\ldots,n+1. \tag{6.6}$$

Therefore, the stability condition (6.2) is satisfied if all eigenvalues (6.6) are strictly positive, in particular we need to assume that there exists a positive constant c_0 independent of τ_ℓ such that

$$3+n-\sqrt{\sum_{i=1}^{n+1}\frac{1}{\hat{h}_i^2}\cdot\sum_{i=1}^{n+1}\hat{h}_i^2} \;\geq\; c_0 \tag{6.7}$$

is satisfied. In fact, (6.7) is a mesh condition on the finite element mesh under consideration, in particular the local behaviour of the mesh size in adaptive triangulations. Note that in case of a globally uniform mesh we have $\hat{h}_i = h$ for all i and therefore (6.7) holds with $c_0 = 2$. Moreover, if a finite element mesh Ω_h is given, the mesh condition (6.7) and therefore the stability assumption (4.3) can be checked in an explicit form and, if necessary, improved by local refinements.
To demonstrate the applicability of the mesh condition (6.7) we consider an adaptive finite element mesh as shown in Figure 1. This mesh was generated by an adaptive algorithm as described in [9].
In Table 1 we give the computational results to find c_0 as minimal value over all elements τ_ℓ for the adaptive hierarchy of the finite element triangulation over all mesh levels $L = 0,\ldots,9$; and M is the number of all finite element nodes.
While in this paper we considered only the case of piecewise linear basis functions, the same approach may be used for more general definitions of V_h. Note that also the case $V_h \subset H_0^1(\Omega)$ with basis functions vanishing along the boundary $\partial\Omega$ can be considered with slight modifications only. In this case we consider the index set $I(k)$ only for nodes x_k associated with

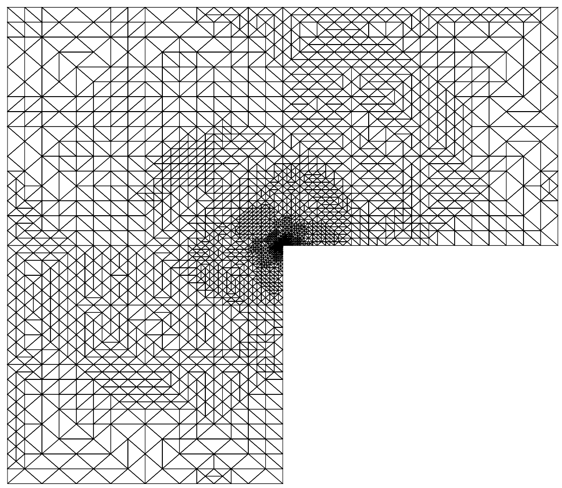

Figure 1: Adaptive finite element triangulation.

L	0	1	2	3	4	5	6	7	8	9
M	8	17	28	53	87	155	291	532	1034	2003
c_0	2.00	1.93	1.59	1.52	1.52	1.66	1.61	1.09	1.49	1.50

Table 1: Computational results for c_0.

a basis function $\varphi_k \in V_h$. Then all proofs given above remain to be valid. Note that the dimension of the local matrices considered in this section decreases when removing the corresponding rows and columns. Moreover, the resulting stability conditions are weaker than in the case that an element has no boundary nodes.

References

[1] J. H. Bramble, J. E. Pasciak, P. S. Vassilevski, Computational scales of Sobolev norms with applications to preconditioning. Math. Comp. (1999) to appear.

[2] J. H. Bramble, J. Xu, Some estimates for a weighted L^2 projection. Math. Comp. 56 (1991) 463–476.

[3] P. G. Ciarlet, The Finite Element Method for Elliptic Problems. North–Holland, 1978.

[4] P. Clement, Approximation by finite element functions using local regularization. RAIRO Anal. Numer. R–2 (1975) 77–84.

[5] M. Crouzeix, V. Thomeé, The stability in L_p and W_p^1 of the L_2–projection onto finite element function spaces. Math. Comp. 48 (1987) 521–532.

[6] W. McLean, O. Steinbach, Boundary element preconditioners for a hypersingular integral equation on a curve. Submitted, 1999.

[7] L. R. Scott, S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions. Math. Comp. 54 (1990) 483–493.

[8] H. Schulz, O. Steinbach, A new a posteriori error estimator in direct boundary element methods. The Neumann problem. Submitted, 1999.

[9] O. Steinbach, Adaptive finite element–boundary element solution of boundary value problems. J. Comput. Appl. Math. 106 (1999) 307–316.

[10] O. Steinbach, On a hybrid boundary element method. Numer. Math., accepted, 1998.

[11] O. Steinbach, W. L. Wendland, The construction of some efficient preconditioners in the boundary element method. Adv. Comput. Math. 9 (1998) 191–216.

[12] L. B. Wahlbin, Superconvergence in Galerkin Finite Element Methods. Lecture Notes in Mathematics 1605, Springer, Berlin, 1995.