REGULARITY ESTIMATES FOR SOLUTIONS OF THE EQUATIONS OF LINEAR ELASTICITY IN CONVEX PLANE POLYGONAL DOMAIN

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This paper is dedicated to Lawrence E. Payne on the occasion of his 80th birthday

ABSTRACT. The Dirichlet problem for the plane elasticity problem on a convex polygonal domain is considered and it is proved that for data in L^2 the H^2 regularity estimate holds with constants independent of the Lamé coefficients.

1. INTRODUCTION

A regularity estimate for the Stokes problem on convex polygonal domains was proved by Kellogg and Osborn in [8]. Shift estimates for the biharmonic Dirichlet problem on polygonal domains in terms of fractional Sobolev norms are proved for example in [2], [4]. Based on the results from the biharmonic Dirichlet problem we will see in the next section that if the data for the Stokes problem are smoother than L^2, then the solution $(\mathbf{u}, p)$ of the Stokes problem on a convex polygonal domain belongs to a space smoother than $H^2 \times H^1$ and a corresponding shift estimate holds. In the third section a result of Arnold, Scott and Vogelius [1] concerning regular inversion of the divergence operator is used to reduce the elasticity problem to that of the Stokes problem. This is combined with the regularity estimate obtained for the Stokes problem in order to get a regularity estimate for the Dirichlet plane elasticity with constants independent of the Lamé coefficients.

2. A SHIFT THEOREM FOR THE STOKES PROBLEM ON POLYGONAL DOMAINS.

Let Ω be a polygonal domain in R^2 with boundary $\partial\Omega$ and let ω be the measure of the largest angle of $\partial\Omega$. First, we review a shift estimate for the biharmonic problem. Find u such that

$$\begin{cases} \Delta^2 u = f \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

We associate to (2.1) the characteristic equation (cf. [7])

$$\sin^2(z\omega) = z^2 \sin^2 \omega. \tag{2.2}$$

Then, according to [2] or [4], there exists a $\gamma_0 \in (0, 1)$,

$$\gamma_0 := \min\{Re(z) \in (1, 2), \text{ z is solution of the equation (2.2)}\} - 1, \tag{2.3}$$

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such that

$$\|u\|_{H^{3+\gamma}} \le c\|f\|_{H^{-1+\gamma}}, \qquad \text{for all } f \in H^{-1+\gamma}(\Omega), \quad 0 < \gamma < \gamma_0. \tag{2.4}$$

Next, we consider the steady-state Stokes problem in the velocity-pressure formulation with $\mathbf{F} \in (H^\gamma)^2$. The spaces H^s are standard and we omit the domain Ω in the notation. The Stokes problem is the following. Find the vector-valued function $\mathbf{u}$ and the scalar-valued function p satisfying

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{F} & \text{in } \ \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \ \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \\ \int_\Omega p = 0 & . \end{cases} \tag{2.5}$$

These equations are to be considered in the standard weak sense. According to [8], we have that $\mathbf{u}$ is in $(H^2)^2$ with $\mathbf{F} \in (L^2)^2$. Since $\nabla \cdot \mathbf{u} = 0 \quad$ in $\ \Omega$ and $\mathbf{u} = 0 \quad$ on $\partial\Omega$, one can find $w \in H_0^2$ (see for example I.3.1 in [5]) such that

$$\mathbf{u} = (u_1, u_2) = \mathbf{curl}\ w := \left(\frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1}\right).$$

If we substitute $\mathbf{u} = \mathbf{curl}\ w$ in (2.5), we get

$$\begin{cases} -\Delta(\frac{\partial w}{\partial x_2}) + \frac{\partial p}{\partial x_1} = f_1 & \text{in } \ \Omega, \\ \Delta(\frac{\partial w}{\partial x_1}) + \frac{\partial p}{\partial x_2} = f_2 & \text{in } \ \Omega, \end{cases} \tag{2.6}$$

where $\mathbf{F} = (f_1, f_2)$. Next, we apply the differential operators $-\frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial x_1}$ to the first and second equations of (2.6) respectively and sum up the two new equations. Thus, we have that $w \in H_0^2$ and

$$\Delta^2 w = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \quad \text{in } \ \Omega.$$

Consequently, for a fixed $\gamma \in (0, \gamma_0)$, from (2.4), we have that

$$\|w\|_{H^{3+\gamma}} \le c \left\| \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right\|_{H^{-1+\gamma}},$$

where c is a constant independent of $\mathbf{F}$. It follows that

$$\|\mathbf{u}\|_{(H^{2+\gamma})^2} \le c\|\mathbf{F}\|_{(H^\gamma)^2}, \qquad \text{for all } \mathbf{F} \in (H^\gamma(\Omega))^2.$$

From the first part of (2.5) we have $\nabla p = \Delta \mathbf{u} + \mathbf{F}$. Hence,

$$\|\nabla p\|_{(H^\gamma)^2} \le \|\Delta \mathbf{u}\|_{(H^\gamma)^2} + \|\mathbf{F}\|_{(H^\gamma)^2} \le \|\mathbf{u}\|_{(H^{2+\gamma})^2} + \|\mathbf{F}\|_{(H^\gamma)^2} \le c\|\mathbf{F}\|_{(H^\gamma)^2}.$$

We conclude this section with the following theorem.

Theorem 2.1. *Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with ω the measure of the largest angle. Let γ_0 be defined by* (2.3) *and let $(\mathbf{u}, p)$ be the solution of* (2.5)*. Then for any $\gamma \in (0, \gamma_0)$ there exist a constant c such that*

$$\|\mathbf{u}\|_{(H^{2+\gamma})^2} + \|p\|_{H^{1+\gamma}} \le c\|\mathbf{F}\|_{(H^\gamma)^2} \qquad \textit{for all } \mathbf{F} \in (H^\gamma)^2. \tag{2.7}$$

3. Uniform regularity for the planar elasticity problem

Let Ω be a convex polygonal domain in R^2 with boundary $\partial\Omega$ as in the previous section. The pure displacement problem for the planar elasticity system is given by

$$\begin{cases} \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla(\nabla\cdot\mathbf{u}) = \mathbf{F} & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where μ and λ are the Lamé coefficients (cf. [6]). We assume that $\mu \in [\mu_1, \mu_2]$ with μ_1, μ_2 fixed positive constants and that $\lambda \geq 0$. The main purpose of this section is to prove that

$$\|\mathbf{u}\|_{(H^2)^2} \leq c\|\mathbf{F}\|_{(L^2)^2}, \qquad \text{for all } \mathbf{F} \in (L^2)^2, \tag{3.2}$$

with a constant c independent of the Lamé coefficients. Since μ is restricted to the compact interval $[\mu_1, \mu_2]$ we can divide the first equation in (3.1) by μ and set $\tilde{\lambda} = \frac{\mu+\lambda}{\mu}$. Thus we have reduced the problem to proving the regularity estimate (3.2) with c independent of $\tilde{\lambda}$, where $\mathbf{u}$ is the (weak) solution of

$$\begin{cases} \Delta\mathbf{u} + \tilde{\lambda}\nabla(\nabla\cdot\mathbf{u}) = \mathbf{F} & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Now from [6] the estimate (3.2) holds for $1 \leq \tilde{\lambda} \leq \lambda_0 < \infty$ with c uniform for each fixed λ_0. Hence it remains to prove (3.2) for $\lambda_0 < \tilde{\lambda}$ with c depending only on λ_0 and Ω.

For any value of $\tilde{\lambda}$, using the Kondratiev method [9] (see also 4.6 in [7]), we have that for a small enough positive number γ the solution $\mathbf{u}$ of (3.3) belongs to $(H^{2+\gamma})^2$, provided $\mathbf{F} \in (H^\gamma)^2$. Using Theorem 3.1 of [1] there exists a function $\mathbf{w} \in (H^{2+\gamma})^2 \cap (H_0^1)^2$ with the following properties

$$\nabla\cdot\mathbf{w} = \nabla\cdot\mathbf{u},$$

and

$$\|\mathbf{w}\|_{(H^{2+\gamma})^2} \leq c\|\nabla\cdot\mathbf{u}\|_{H^{1+\gamma}}, \tag{3.4}$$

with c independent of $\mathbf{u}$. Next, let us set $\mathbf{v} := \mathbf{w} - \mathbf{u}$. Thus,

$$\begin{cases} -\Delta\mathbf{v} + \tilde{\lambda}\nabla(\nabla\cdot\mathbf{u}) = \mathbf{F} - \Delta\mathbf{w} & \text{in } \Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

If we denote $p := \tilde{\lambda}\nabla\cdot\mathbf{u}$, then

$$\begin{cases} -\Delta\mathbf{v} + \nabla p = \mathbf{F} - \Delta\mathbf{w} & \text{in } \Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

From the regularity for the Stokes problem, Theorem 2.1, we get

$$\|\mathbf{v}\|_{(H^{2+\gamma})^2} + \tilde{\lambda}\|\nabla\cdot\mathbf{u}\|_{H^{1+\gamma}} \leq c(\|\mathbf{F}\|_{(H^\gamma)^2} + \|\Delta\mathbf{w}\|_{(H^\gamma)^2}), \qquad \text{for all } \mathbf{F} \in (H^\gamma)^2. \tag{3.7}$$

From (3.4) we have, for another constant c, that

$$\|\mathbf{v}\|_{(H^{2+\gamma})^2} + \tilde{\lambda}\|\nabla\cdot\mathbf{u}\|_{H^{1+\gamma}} \leq c(\|\mathbf{F}\|_{(H^\gamma)^2} + \|\nabla\cdot\mathbf{u}\|_{H^{1+\gamma}}), \qquad \text{for all } \mathbf{F} \in (H^\gamma)^2.$$

Hence, using again (3.4), we conclude that for a constant c_1, independent of $\tilde{\lambda}$,

$$\|\mathbf{u}\|_{(H^{2+\gamma})^2} \leq c\|\mathbf{F}\|_{(H^\gamma)^2} + (c_1 - \tilde{\lambda})\|\nabla \cdot \mathbf{u}\|_{H^{1+\gamma}}, \qquad \text{for all } \mathbf{F} \in (H^\gamma)^2. \tag{3.8}$$

Thus, for $\tilde{\lambda} > c_1$ we obtain

$$\|\mathbf{u}\|_{(H^{2+\gamma})^2} \leq c\|\mathbf{F}\|_{(H^\gamma)^2}, \qquad \text{for all } \mathbf{F} \in (H^\gamma)^2. \tag{3.9}$$

It is well known that the variational solution of (3.1) satisfies

$$\|\mathbf{u}\|_{(H^1)^2} \leq c_2\|\mathbf{F}\|_{(H^{-1})^2}, \qquad \text{for all } \mathbf{F} \in (H^{-1})^2, \tag{3.10}$$

with c_2 independent of $\tilde{\lambda}$. By interpolation (cf. [10, 11]) we have that (3.2) holds with a constant c independent of $\tilde{\lambda}$ for $\tilde{\lambda} > c_1$. Since $\tilde{\lambda} \to \lambda$ is continuous on $(1, \infty)$ we can conclude the following result.

Theorem 3.1. *Let Ω be a convex polygonal domain in $\mathbb{R}^2$ and let $\mathbf{u}$ be the solution of* (3.1)*. Then the shift estimate* (3.2) *holds with a constant c independent of the Lamé coefficients, for $\mu \in [\mu_1, \mu_2]$ and $\lambda \in [0, \infty)$.*

Remark 3.1. *This result was stated in* [3] *and used in certain finite element estimates. It was similarly used in* [12].

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