

A Locking-Free Finite Element Method for Naghdi Shells

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Abstract

In this paper a locking-free method, using mixed finite elements, is introduced to approximate the solution of Naghdi shell problems with small parameter t , the thickness of the shell. The approach of Arnold and Brezzi [1] is employed with some important changes. Instead of proving the discrete Inf-Sup condition for arbitrary geometric coefficients, which does not seem possible, we prove a weaker stability condition for smooth enough geometrically dependent coefficients, which is sufficient for deducing the optimal error estimate as long as h^2/t is uniformly bounded. For extremely small t , we can relax this condition either using a larger bubble space or stabilizing the problem by replacing t^2 by $t^2 + h^4$. In either case an optimal error estimate still holds.

Key words: Shell, locking, mixed finite element

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1 Introduction

Finite element solutions of bending dominated thin shell models suffer both shear and membrane locking. The special case of Reissner-Mindlin plates, where only shear locking is present, has been well understood and successfully overcome in several papers [12][3][13][15][16] using mixed finite element methods and by Bramble and Sun [9] using a discrete negative norm least squares method. For shells, the geometry and the membrane strain tensor make the problem far more complicated. There has been some work dealing with special cases, such as that of a cylinder [18]. But, as far as we know, there has not been any method mathematically proven to be locking free for the general case.

The paper of Arnold and Brezzi[1] briefly surveyed the literature, treated the Naghdi shell model as an abstract saddle point problem, gave a sequence of mixed finite element methods with triangle elements and proved an error estimate independent of t under the restrictive assumption that all the geometric coefficients are constant on each element. They conjectured that the assumption is only needed by their method for proving the discrete Inf-Sup condition. Suri [20] used an hp finite element method on parallelogram elements and he increased the order of approximation polynomials to allow polynomial geometric coefficients, which improved the result of Arnold and Brezzi with a certain added cost. It should be mentioned that all the geometric coefficients cannot be polynomials at the same time, unless they are all constants. They could, however, be approximated by polynomials. Stenberg and Chapelle [19] used stabilization techniques to avoid the discrete inf-sup condition. But the stabilization for the complicated Koiter shell model seems to be too expensive, and the analysis in the paper involved full regularity assumption which is by itself an open problem with some well-known negative results [2].

We believe that the approach of Arnold and Brezzi successfully avoided the trouble, which could be created by the membrane strain term, by introducing a seminorm instead of a norm for the new unknowns of shear and membrane stresses. Thus we also take this approach. In this paper, we define new unknowns for shear and membrane stresses in a slightly different way, which significantly eases the difficulty of satisfying the discrete Inf-Sup condition. At the lowest order, piecewise constant elements are used for these new unknowns, continuous piecewise linear functions plus the bubble space B_3 is used for the rotations, and continuous piecewise polynomials of degree 2 plus the bubble space B_3 is used for the displacements. We prove a weaker stability result on the subspaces for smooth enough geometric coefficients. From this result, we know that when the coefficients are in $C^{0,1}$ and $h^2 < Ct$, the method is locking-free. We don't think that $h^2 < Ct$ is a very restrictive condition, but if in any case, we have two ways to relax it. First, by using a larger bubble space B_m for the displacements, we only need $h^{m-1} < Ct$. Alternatively, we can stabilize the problem by replacing t^2 by $t^2 + h^4$. A number of papers used this sort of stabilization for Reissner-Mindlin plates (e.g.[13]). L^2 regularity of the shear and membrane stresses is needed to estimate the error introduced by this perturbation; there are no theoretical results proving this regularity on shells.

The rest of the paper is organized as follows. Section 2 describes the Naghdi shell model and some necessary geometric notation. Section 3 introduces the locking-free reformulation with shear and membrane stresses as new unknowns. Section 4 presents the abstract error estimate with our weaker stability assumption. In Section 5 the mixed finite element spaces are defined and the stability is proven under the condition $h^2 < Ct$. Section 6 contains the two ways of relaxing the condition. Section 7 gives a brief and abstract description

of the widely used reduced integration technique, followed by its application to our scheme.

2 Naghdi shell model and some geometry notations

For the convenience of the reader, we basically use the same notation used by Arnold and Brezzi in [1]. The Greek indices range over 1 and 2, while the Latin indices range from 1 to 3. When indices are repeated in a single term, this means that we sum over that index. Overarrows are used to denote vectors in \mathbb{R}^3 , undertildes for vectors in \mathbb{R}^2 , and double undertildes for 2×2 symmetric tensors.

Suppose that the midsurface S of the shell under concern is parametrized by a single chart with a smooth mapping $\vec{r} : \Omega \subset \mathbb{R}^2 \longrightarrow S \subset \mathbb{R}^3$.

$$\vec{a}_1 = \frac{\partial \vec{r}}{\partial x^1}, \quad \vec{a}_2 = \frac{\partial \vec{r}}{\partial x^2}, \quad \vec{a}_3 = \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|}.$$

\vec{a}_1 and \vec{a}_2 are linearly independent at each point of S , and the shell occupies the region

$$\{\vec{r}(x) + x^3 \vec{a}_3(x) | x \in \Omega, -t/2 < x^3 < t/2\} \subset \mathbb{R}^3.$$

Let $a_{\alpha\beta}$ and $b_{\alpha\beta}$ be the first and second fundamental forms of the midsurface under the parametrization \vec{r} ,

$$a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta, \quad b_{\alpha\beta} = -\vec{a}_\alpha \cdot \frac{\partial \vec{a}_3}{\partial x^\beta}.$$

Let $(a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}$, $\vec{a}^\alpha = a^{\alpha\beta} \vec{a}_\beta$, $\vec{a}^3 = \vec{a}_3$, $b_\gamma^\alpha = a^{\alpha\beta} b_{\beta\gamma}$, $a = \det(a_{\alpha\beta})$, and the Christoffel symbol $\Gamma_{\alpha\beta}^\delta = \vec{a}^\delta \cdot \frac{\partial \vec{a}_\beta}{\partial x^\alpha}$. For any integrable function $f : S \longrightarrow \mathbb{R}$, we have $\int_S f = \int_\Omega f(\vec{r}) \sqrt{a} d\tilde{x}$.

Define $a^{\alpha\beta\gamma\delta} = \frac{E}{1-\nu^2} (a^{\alpha\gamma} a^{\beta\delta} + \nu a^{\alpha\beta} a^{\gamma\delta})$, where $E > 0$ and $0 < \nu < 1/2$ are the Young's modulus and Poisson ratio of the material respectively.

Let $\vec{u} : \tilde{x} \in \Omega \longmapsto \vec{u}(\tilde{x}) \in \mathbb{R}^3$ be the displacement of the shell at $r(\tilde{x}) \in S$, $\theta : \tilde{x} \in \Omega \longmapsto \theta(\tilde{x}) \in \mathbb{R}^2$ be the rotation of the fiber in the shell perpendicular to the midsurface at $r(\tilde{x}) \in S$. $\vec{u}(\tilde{x}) = u_i(\tilde{x}) \vec{a}^i(\tilde{x})$, $\theta(\tilde{x}) = \theta_\alpha(\tilde{x}) \vec{a}^\alpha(\tilde{x})$. For $u_i, \theta_\alpha \in H^1(\Omega)$, define the bending curvature tensor $\tilde{\Upsilon}$, the transverse shear strain tensor $\tilde{\Phi}$, and the membrane strain tensor $\tilde{\Lambda}$ as follows:

$$\Upsilon_{\alpha\beta}(\vec{u}, \tilde{\theta}) = \frac{1}{2}[\theta_{\alpha,\beta} + \theta_{\beta,\alpha} - b_{\alpha}^{\gamma}(u_{\gamma,\beta} - b_{\gamma\beta}u_3 - \Gamma_{\gamma\beta}^{\delta}u_{\delta}) - b_{\beta}^{\gamma}(u_{\gamma,\alpha} - b_{\gamma\alpha}u_3 - \Gamma_{\gamma\alpha}^{\delta}u_{\delta})] - \Gamma_{\alpha\beta}^{\delta}\theta_{\delta}, \quad (2.1)$$

$$\Phi_{\alpha}(\vec{u}, \tilde{\theta}) = \theta_{\alpha} + u_{3,\alpha} + b_{\alpha}^{\gamma}u_{\gamma}, \quad (2.2)$$

$$\Lambda_{\alpha\beta}(\vec{u}) = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) - \Gamma_{\alpha\beta}^{\delta}u_{\delta} - b_{\alpha\beta}u_3. \quad (2.3)$$

Let

$$H_{\Gamma_d}^1 = \{v \in H^1(\Omega) | v(x) = 0, \forall x \in \Gamma_d \subset \partial\Omega\},$$

$$V = \{(\vec{v}, \tilde{\psi}) | v_i, \psi_{\alpha} \in H_{\Gamma_d}^1\},$$

$$\|\vec{v}, \tilde{\psi}\|_V^2 = \|\vec{v}\|_1^2 + \|\tilde{\psi}\|_1^2,$$

where Γ_d has positive measure.

The bending energy E^B , shearing energy E^S , membrane energy E^M and the work W^L done by the load \vec{f} are

$$E^B = \frac{t^3}{24} \int_{\Omega} a^{\alpha\beta\gamma\delta} \Upsilon_{\alpha\beta}(\vec{v}, \tilde{\psi}) \Upsilon_{\gamma\delta}(\vec{v}, \tilde{\psi}) \sqrt{a} dx_{\tilde{\sim}},$$

$$E^S = \frac{t}{2} \int_{\Omega} \frac{E}{2(1+\nu)} a^{\alpha\beta} \Phi_{\alpha}(\vec{v}, \tilde{\psi}) \Phi_{\beta}(\vec{v}, \tilde{\psi}) \sqrt{a} dx_{\tilde{\sim}},$$

$$E^M = \frac{t}{2} \int_{\Omega} a^{\alpha\beta\gamma\delta} \Lambda_{\alpha\beta}(\vec{v}) \Lambda_{\gamma\delta}(\vec{v}) \sqrt{a} dx_{\tilde{\sim}},$$

$$W^L = t^3 \int_{\Omega} f^i v_i \sqrt{a} dx_{\tilde{\sim}}.$$

So the Naghdi shell problem is to find $(\vec{u}, \tilde{\theta}) \in V$ as the minimizer of the total energy

$$E(\vec{v}, \tilde{\psi}) = E^B + E^S + E^M - W^L. \quad (2.4)$$

The equivalent variational formulation is to find $(\vec{u}, \tilde{\theta}) \in V$ such that

$$\begin{aligned} & \frac{1}{12} \int_{\Omega} a^{\alpha\beta\gamma\delta} \Upsilon_{\alpha\beta}(\vec{u}, \tilde{\theta}) \Upsilon_{\gamma\delta}(\vec{v}, \tilde{\psi}) \sqrt{a} dx_{\tilde{\sim}} \\ & + t^{-2} \int_{\Omega} \frac{E}{2(1+\nu)} a^{\alpha\beta} \Phi_{\alpha}(\vec{u}, \tilde{\theta}) \Phi_{\beta}(\vec{v}, \tilde{\psi}) \sqrt{a} dx_{\tilde{\sim}} \\ & + t^{-2} \int_{\Omega} a^{\alpha\beta\gamma\delta} \Lambda_{\alpha\beta}(\vec{u}) \Lambda_{\gamma\delta}(\vec{v}) \sqrt{a} dx_{\tilde{\sim}} \\ & = \int_{\Omega} f^i v_i \sqrt{a} dx_{\tilde{\sim}}, \quad \forall (\vec{v}, \tilde{\psi}) \in V. \end{aligned} \quad (2.5)$$

3 Locking-free reformulation

The reason for locking is well understood[6]. The general idea of avoiding locking in plates and shells problems is to introduce a new unknown for a term which looks like $\epsilon^{-2}(Lw - Dz)$, where L, D are differential operators, w, z are functions, ϵ is a small parameter. Here we define two new unknowns for the shear stress $\underset{\sim}{\phi}$ and the membrane stress $\underset{\approx}{\lambda}$ by

$$\phi^\beta = \epsilon^{-2} \frac{E}{2(1+\nu)} a^{\alpha\beta} \Phi_\alpha(\vec{u}, \underset{\sim}{\theta}) \sqrt{a}, \quad (3.6)$$

and

$$\lambda^{\gamma\delta} = \epsilon^{-2} a^{\alpha\beta\gamma\delta} \Lambda_{\alpha\beta}(\vec{u}) \sqrt{a}. \quad (3.7)$$

The purpose of including $(a^{\alpha\beta})$ in $\underset{\sim}{\phi}$ and $(a^{\alpha\beta\gamma\delta})$ in $\underset{\approx}{\lambda}$ is to remove the geometric coefficients from certain terms as much as possible, so that there will be less trouble in satisfying a discrete stability condition.

For any fixed constant $c_0 > 0$, let

$$\begin{aligned} A(\vec{u}, \underset{\sim}{\theta}; \vec{v}, \underset{\sim}{\psi}) &= \frac{1}{12} \int_{\Omega} a^{\alpha\beta\gamma\delta} \Upsilon_{\alpha\beta}(\vec{u}, \underset{\sim}{\theta}) \Upsilon_{\gamma\delta}(\vec{v}, \underset{\sim}{\psi}) \sqrt{a} dx \\ &+ c_0 \int_{\Omega} \frac{E}{2(1+\nu)} a^{\alpha\beta} \Phi_\alpha(\vec{u}, \underset{\sim}{\theta}) \Phi_\beta(\vec{v}, \underset{\sim}{\psi}) \sqrt{a} dx \\ &+ c_0 \int_{\Omega} a^{\alpha\beta\gamma\delta} \Lambda_{\alpha\beta}(\vec{u}) \Lambda_{\gamma\delta}(\vec{v}) \sqrt{a} dx. \end{aligned} \quad (3.8)$$

$A(\cdot, \cdot; \cdot, \cdot)$ is coercive on V , as Bernadou and Ciarlet proved in [7]. Let

$$B_s(\vec{v}, \underset{\sim}{\psi}; \underset{\sim}{\eta}) = \int_{\Omega} \Phi_\beta(\vec{v}, \underset{\sim}{\psi}) \eta^\beta dx, \quad (3.9)$$

$$C_s(\underset{\sim}{\phi}, \underset{\sim}{\eta}) = \int_{\Omega} \frac{2(1+\nu)}{E} a_{\alpha\beta} \phi^\alpha \eta^\beta \frac{1}{\sqrt{a}} dx, \quad (3.10)$$

$$B_m(\vec{v}, \underset{\approx}{\chi}) = \int_{\Omega} \Lambda_{\alpha\beta}(\vec{v}) \chi^{\alpha\beta} dx, \quad (3.11)$$

$$C_m(\underset{\approx}{\lambda}, \underset{\approx}{\chi}) = \int_{\Omega} a_{\alpha\beta\gamma\delta} \lambda^{\alpha\beta} \chi^{\gamma\delta} \frac{1}{\sqrt{a}} dx. \quad (3.12)$$

Here $(a_{\alpha\beta\gamma\delta})$ is the inverse of the 4×4 tensor $(a^{\alpha\beta\gamma\delta})$. One can easily verify that

$$a_{\alpha\beta\gamma\delta} = \frac{1-\nu^2}{E} (a_{\alpha\gamma} a_{\beta\delta} - \frac{\nu}{1+2\nu} a_{\alpha\beta} a_{\gamma\delta}).$$

Then problem (2.5) is equivalent to that of finding $(\vec{u}, \underline{\theta}) \in V$, $\underline{\phi} \in W_s = \{\eta | \eta^\alpha \in L^2(\Omega)\}$ and $\underline{\lambda} \in W_m = \{\chi | \chi^{\alpha\beta} \in L^2(\Omega)\}$ such that

$$\begin{aligned} A(\vec{u}, \underline{\theta}; \vec{v}, \underline{\psi}) + B_s(\vec{v}, \underline{\psi}; \underline{\phi}) + B_m(\vec{v}, \underline{\lambda}) &= F(\vec{v}), \quad \forall (\vec{v}, \underline{\psi}) \in V, \\ B_s(\vec{u}, \underline{\theta}; \underline{\eta}) - \epsilon^2 C_s(\underline{\phi}, \underline{\eta}) &= 0, \quad \forall \underline{\eta} \in W_s, \\ B_m(\vec{u}, \underline{\lambda}) - \epsilon^2 C_m(\underline{\lambda}, \underline{\chi}) &= 0, \quad \forall \underline{\chi} \in W_m, \end{aligned} \quad (3.13)$$

where $\epsilon^2 = \frac{t^2}{1-c_0 t^2}$, $F(\vec{v}) = \int_{\Omega} f^i v_i \sqrt{a} dx$.

While (3.13) is convenient for implementation, to analyze the stability and estimate the error, it is more convenient to define

$$B(\vec{v}, \underline{\psi}; \underline{\eta}, \underline{\chi}) = B_s(\vec{v}, \underline{\psi}; \underline{\eta}) + B_m(\vec{v}, \underline{\chi}),$$

$$C(\underline{\phi}, \underline{\lambda}; \underline{\eta}, \underline{\chi}) = C_s(\underline{\phi}, \underline{\eta}) + C_m(\underline{\lambda}, \underline{\chi}),$$

$W = W_s \times W_m$ and consider the more abstract version of the problem: Find $(\vec{u}, \underline{\theta}) \in V$ and $(\underline{\phi}, \underline{\lambda}) \in W$ such that

$$\begin{aligned} A(\vec{u}, \underline{\theta}; \vec{v}, \underline{\psi}) + B(\vec{v}, \underline{\psi}; \underline{\phi}, \underline{\lambda}) &= F(\vec{v}), \quad \forall (\vec{v}, \underline{\psi}) \in V, \\ B(\vec{u}, \underline{\theta}; \underline{\eta}, \underline{\chi}) - \epsilon^2 C(\underline{\phi}, \underline{\lambda}; \underline{\eta}, \underline{\chi}) &= 0, \quad \forall (\underline{\eta}, \underline{\chi}) \in W. \end{aligned} \quad (3.14)$$

The following seminorm is defined in [1],

$$|\underline{\eta}, \underline{\chi}|_W = \sup_{(\vec{v}, \underline{\psi}) \in V} \frac{B(\vec{v}, \underline{\psi}; \underline{\eta}, \underline{\chi})}{\|\vec{v}, \underline{\psi}\|_V}, \quad \forall (\underline{\eta}, \underline{\chi}) \in W. \quad (3.15)$$

It is easily shown (cf. [1]) that, as a consequence of the Lax-Milgram Theorem, (3.14) has a unique solution and that

$$\|\vec{u}, \underline{\theta}\|_V + |\underline{\phi}, \underline{\lambda}|_W + \epsilon \|\underline{\phi}, \underline{\lambda}\|_{L^2} \leq C \|F\|_{V^*}. \quad (3.16)$$

4 Abstract error estimate with stability assumption

After including $(a^{\alpha\beta\gamma\delta})\sqrt{a}$ and $(a^{\alpha\beta})\sqrt{a}$ into the new unknowns, there are still some geometric coefficients left in the following terms

$$B_s(\vec{v}, \psi; \eta) = \int_{\Omega} \eta^\alpha (\psi_\alpha + v_{3,\alpha} + b_\alpha^\gamma v_\gamma) dx_{\tilde{\sim}}$$

and

$$B_m(\vec{v}, \chi) = \int_{\Omega} \chi^{\alpha\beta} \left[\frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}) - \Gamma_{\alpha\beta}^\delta v_\delta - b_{\alpha\beta} v_3 \right] dx_{\tilde{\sim}}.$$

Therefore it does not seem to be possible to prove a discrete version of the Inf-Sup condition. As a substitute for the Inf-Sup condition, we will establish a slightly weaker stability condition. This weaker version of the stability condition might be useful for other parameter dependent problems, therefore we present it abstractly.

Let V and W be Hilbert spaces, and let the bilinear forms

$$A : V \times V \rightarrow \mathbb{R}, \quad B : V \times W \rightarrow \mathbb{R}, \quad C : W \times W \rightarrow \mathbb{R}$$

and the linear functional $F : V \rightarrow \mathbb{R}$ be continuous. Suppose that A and C are coercive, so the norms induced by A in V and C in W are equivalent to the Hilbert space norms $\|\cdot\|_V$ and $\|\cdot\|_W$ respectively. Define a seminorm on W by

$$|Q|_W = \sup_{Z \in V} \frac{B(Z, Q)}{\|Z\|_V}, \quad Q \in W.$$

Consider a parameter dependent saddle point problem of the following form:
Problem $V - W$: Find $U \in V$ and $P \in W$ such that

$$\begin{aligned} A(U, Z) + B(Z, P) &= F(Z), & \forall Z \in V, \\ B(U, Q) - \epsilon^2 C(P, Q) &= 0, & \forall Q \in W. \end{aligned} \tag{4.17}$$

Consider subspaces $V_h \subset V$ and $W_h \subset W$. An approximate solution of *Problem $V - W$* in $V_h \times W_h$ is given by

Problem $V_h - W_h$: Find $U_h \in V_h$ and $P_h \in W_h$ such that

$$\begin{aligned} A(U_h, Z) + B(Z, P_h) &= F(Z), & \forall Z \in V_h, \\ B(U_h, Q) - \epsilon^2 C(P_h, Q) &= 0, & \forall Q \in W_h. \end{aligned} \tag{4.18}$$

A weakening of the discrete Inf-Sup condition is the following:

A1: There exists $\gamma > 0$ and $\hat{C} > 0$ such that

$$|Q|_W \leq \frac{1}{\gamma} \sup_{Z \in V_h} \frac{B(Z, Q)}{\|Z\|_V} + \hat{C}\epsilon \|Q\|_W, \quad \forall Q \in W_h. \quad (4.19)$$

Under this condition, we have the following theorem.

Theorem 1 *Let (U, P) be the solution of (4.17), (U_h, P_h) be the solution of (4.18). Suppose **A1** holds. Then there is a constant \bar{C} , depending only on γ, \hat{C} , the continuity and coercivity constants of the bilinear forms A and C , such that*

$$\begin{aligned} & \|U - U_h\|_V + |P - P_h|_W + \epsilon \|P - P_h\|_W \\ & \leq \bar{C} \inf_{\substack{P^* \in W_h \\ U^* \in V_h}} (\|U - U^*\|_V + |P - P^*|_W + \epsilon \|P - P^*\|_W). \end{aligned} \quad (4.20)$$

Proof.

The existence and uniqueness of the solutions of both the problems follow from the Lax-Milgram theorem. Subtracting (4.18) from (4.17), we get

$$A(U - U_h, Z) + B(Z, P - P_h) = 0, \quad \forall Z \in V_h,$$

$$B(U - U_h, Q) - \epsilon^2 C(P - P_h, Q) = 0, \quad \forall Q \in W_h.$$

For any $U^* \in V_h$ and $P^* \in W_h$,

$$A(U_h - U^*, Z) + B(Z, P_h - P^*) = A(U - U^*, Z) + B(Z, P - P^*) \quad (4.21)$$

$$B(U_h - U^*, Q) - \epsilon^2 C(P_h - P^*, Q) = B(U - U^*, Q) - \epsilon^2 C(P - P^*, Q) \quad (4.22)$$

for all $Z \in V_h, Q \in W_h$. Now subtracting (4.22) from (4.21) with $Z = U_h - U^*$ and $Q = P_h - P^*$, using the coercivity of the bilinear forms A and C and the and continuity of the forms A, B and C , we have

$$\begin{aligned} & \|U_h - U^*\|_V^2 + \epsilon^2 \|P_h - P^*\|_{L^2}^2 \\ & \leq C(\|U - U^*\|_V \|U_h - U^*\|_V + \|U_h - U^*\|_V |P - P^*|_W \\ & \quad + \|U - U^*\|_V |P_h - P^*|_W + \epsilon^2 \|P_h - P^*\|_{L^2} \|P - P^*\|_{L^2}). \end{aligned} \quad (4.23)$$

Follow (4.19) and (4.21),

$$\begin{aligned}
|P_h - P^*|_W &\leq \frac{1}{\gamma} \sup_{Z \in V_h} \frac{B(Z, P_h - P^*)}{\|Z\|_V} + \hat{C}\epsilon \|P_h - P^*\|_{L^2} \\
&\leq C(\|U_h - U^*\|_V + \|U - U^*\|_V + |P - P^*|_W + \epsilon \|P_h - P^*\|_{L^2}). \quad (4.24)
\end{aligned}$$

Now (4.20) is a consequence of (4.23), (4.24) and triangle inequality, and hence the theorem is proven. \square

This theorem says that (4.18) provides a quasi-optimal approximation. An error estimate will depend on whatever regularity is valid for the exact solution (U, P) of (4.17), and the approximation properties of the spaces V_h and W_h .

5 Mixed finite elements and stability

Consider a polygonal domain Ω triangulated regularly by \mathcal{T}_h , with h the maximal mesh size. Let $\mathcal{P}_k(\tau)$ be the set of polynomials of degree k or less restricted to $\tau \in \mathcal{T}_h$. Let

$$L_k^s(\mathcal{T}_h) = \{p \in H^s(\Omega); p|_\tau \in \mathcal{P}_k(\tau), \forall \tau \in \mathcal{T}_h\},$$

$$B_k(\mathcal{T}_h) = \{p \in H_0^1(\Omega); p|_\tau \in \mathcal{P}_k(\tau) \cap H_0^1(\tau), \forall \tau \in \mathcal{T}_h\},$$

and

$$M_k^m(\mathcal{T}_h) = L_k^1(\mathcal{T}_h) \cap H_{\Gamma_d}^1 + B_m(\mathcal{T}_h).$$

We use exactly the same pairs of finite element spaces which Arnold and Brezzi [1] use for Naghdi shells, namely, continuous piecewise polynomials augmented by bubbles for the primitive unknowns, discontinuous piecewise polynomials for the new unknowns.

$$(\vec{u}_h, \theta_h) \in V_h = \{(\vec{v}, \psi) \in V | v_i \in M_{k+2}^{k+3}(\mathcal{T}_h), \psi_\alpha \in M_{k+1}^{k+3}(\mathcal{T}_h)\},$$

$$(\phi, \lambda) \in W_h = \{(\eta, \chi) \in W | \eta^\alpha, \chi^{\alpha\beta} \in L_k^0(\mathcal{T}_h)\}.$$

The difference is that our new unknowns are defined differently from those in [1]. The way of defining the new unknowns plays a very important role in the stability analysis.

Lemma 7 of [1] constructed a linear operator $\pi_h : H_{\Gamma_d}^1 \rightarrow M_{k+2}^{k+3}(\mathcal{T}_h)$ which satisfies the following:

$$\|\pi_h w\|_1 \leq C \|w\|_1, \quad \forall w \in H_{\Gamma_d}^1, \quad (5.25)$$

$$\|w - \pi_h w\|_0 \leq Ch \|w\|_1, \quad \forall w \in H_{\Gamma_d}^1, \quad (5.26)$$

$$\int_{\tau} (w - \pi_h w) p \, dx = 0, \quad \forall w \in H_{\Gamma_d}^1, \quad (5.27)$$

and

$$\int_e (w - \pi_h w) p \, ds = 0, \quad \forall w \in H_{\Gamma_d}^1, \quad (5.28)$$

where e is an edge of τ , τ is a triangle of \mathcal{T}_h , p is a function in $L_k^0(\mathcal{T}_h)$. Furthermore, Green's theorem, (5.27) and (5.28) imply that for arbitrary $\tau \in \mathcal{T}_h$ and $p \in L_k^0(\mathcal{T}_h)$,

$$\int_{\tau} (w - \pi_h w)_{,\alpha} p \, dx = 0, \quad \forall w \in H_{\Gamma_d}^1. \quad (5.29)$$

The method used to construct π_h will be seen in the next section, where we construct a slightly different operator. It is easy to see from the construction[1] that one can define a linear operator $\pi_h^* : H_{\Gamma_d}^1 \rightarrow M_{k+1}^{k+3}(\mathcal{T}_h)$, which only satisfies (5.25), (5.26) and (5.27). For the subspace of the rotation $\tilde{\theta}$, one only needs π_h^* in proving the stability condition **A2** later in this section.

Lemma 1 *Let $p \in L_k^0(\mathcal{T}_h)$, $b : \Omega \rightarrow \mathbb{R}$ be Lipschitz continuous on each $\tau \in \mathcal{T}_h$. Then there is a constant C such that*

$$\sup_{w \in H_{\Gamma_d}^1} \frac{(p, b w)_{L^2}}{\|w\|_1} \leq C \sup_{w \in M_{k+2}^{k+3}(\mathcal{T}_h)} \frac{(p, b w)_{L^2}}{\|w\|_1} + Ch^2 \|p\|_{L^2}. \quad (5.30)$$

Proof: Let π_h be given as above. Define $\bar{b} \in L_0^0(\mathcal{T}_h)$ so that $\bar{b}|_{\tau}$ is equal to the average of b on τ for each triangle $\tau \in \mathcal{T}_h$. Consider $w \neq 0$, we have

$$\begin{aligned} (p, b w)_{\tau} &= (p, b \pi_h w)_{\tau} + (p, b (w - \pi_h w))_{\tau} \\ &= (p, b \pi_h w)_{\tau} + (p, (b - \bar{b})(w - \pi_h w))_{\tau}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{(p, b w)_{\Omega}}{\|w\|_1} &\leq C \frac{(p, b \pi_h w)_{\Omega}}{\|\pi_h w\|_1} + \frac{\|p\|_{L^2} \|b - \bar{b}\|_{L^{\infty}} \|w - \pi_h w\|_{L^2}}{\|w\|_1} \\ &\leq C \frac{(p, b \pi_h w)_{\Omega}}{\|\pi_h w\|_1} + Ch^2 \|p\|_{L^2}. \end{aligned}$$

Here we used (5.25), (5.26) and the Lipschitz continuity of b on each element. The lemma now follows. \square

Remark: In Lemma 1, we are only dealing with geometric coefficients (b) multiplying functions (w) in H^1 , so that (5.26) can be used in the inequalities of

the proof. If we had wanted a similar inequality for the derivative of a function w , we would have had to use (5.25) instead of (5.26). Consequently, we would have lost one power of h in the second term of the right hand side of (5.30). But it is not desirable to require that $h < Ct$. By including most of the geometric coefficients in the two new unknowns (3.6) and (3.7), we have avoided this trouble.

We are now ready to establish stability for $V_h \times W_h$ and the shell problem.

Lemma 2 *Assume that $b_{\alpha\beta}$, $\Gamma_{\alpha\beta}^\delta$ and b_α^δ are Lipschitz continuous. For $(\eta, \chi) \in W_h$,*

$$\begin{aligned} |\eta, \chi|_W &= \sup_{(\vec{v}, \psi) \in V} \frac{B(\vec{v}, \psi; \eta, \chi)}{\|\vec{v}, \psi\|_V} \\ &\leq C \sup_{(\vec{v}, \psi) \in V_h} \frac{B(\vec{v}, \psi; \eta, \chi)}{\|\vec{v}, \psi\|_V} + Ch^2 \|\eta, \chi\|_{L^2}. \end{aligned} \quad (5.31)$$

Proof:

Define $\Pi_h : V \rightarrow V_h$ by

$$\Pi_h(\vec{v}, \psi) = (\pi_h v_1, \pi_h v_2, \pi_h v_3, \pi_h^* \psi_1, \pi_h^* \psi_2). \quad (5.32)$$

It is obvious that $\|\Pi_h(\vec{v}, \psi)\|_V \leq C \|\vec{v}, \psi\|_V$. Using the same argument as in the proof of Lemma 1, we have, for any $(\vec{v}, \psi) \in V$,

$$\begin{aligned} \frac{B(\vec{v}, \psi; \eta, \chi)}{\|\vec{v}, \psi\|_V} &= \frac{B(\Pi_h(\vec{v}, \psi); \eta, \chi)}{\|\vec{v}, \psi\|_V} + \frac{(\eta^\beta, b_\beta^\delta (v_\delta - \pi_h v_\delta))_{L^2}}{\|\vec{v}, \psi\|_V} \\ &\quad + \frac{(\chi^{\alpha\beta}, \Gamma_{\alpha\beta}^\delta (v_\delta - \pi_h v_\delta) + b_{\alpha\beta} (v_3 - \pi_h v_3))_{L^2}}{\|\vec{v}, \psi\|_V} \\ &\leq C \frac{B(\Pi_h(\vec{v}, \psi); \eta, \chi)}{\|\Pi_h(\vec{v}, \psi)\|_V} + \frac{(\eta^\beta, (b_\beta^\delta - \bar{b}_\beta^\delta) (v_\delta - \pi_h v_\delta))_{L^2}}{\|\vec{v}, \psi\|_V} \\ &\quad + \frac{(\chi^{\alpha\beta}, (\Gamma_{\alpha\beta}^\delta - \bar{\Gamma}_{\alpha\beta}^\delta) (v_\delta - \pi_h v_\delta) + (b_{\alpha\beta} - \bar{b}_{\alpha\beta}) (v_3 - \pi_h v_3))_{L^2}}{\|\vec{v}, \psi\|_V} \\ &\leq C \sup_{(\vec{v}, \psi) \in V_h} \frac{B(\vec{v}, \psi; \eta, \chi)}{\|\vec{v}, \psi\|_V} + Ch^2 \|\eta, \chi\|_{L^2}. \end{aligned}$$

Here we used (5.26), (5.27), (5.29) and the Lipschitz continuity of $b_{\alpha\beta}$, $\Gamma_{\alpha\beta}^\delta$ and b_α^δ . $\bar{b}_{\alpha\beta}$, $\bar{\Gamma}_{\alpha\beta}^\delta$ and \bar{b}_α^δ are in $L_0^0(\mathcal{T}_h)$. On each $\tau \in \mathcal{T}_h$, $\bar{b}_{\alpha\beta}$, $\bar{\Gamma}_{\alpha\beta}^\delta$ and \bar{b}_α^δ are equal to the mean values of $b_{\alpha\beta}$, $\Gamma_{\alpha\beta}^\delta$ and b_α^δ respectively. \square

In the case of our Naghdi shell problem, the stability condition (4.19) will be the following.

A2: There exists $\gamma_s > 0$ and $\hat{C} > 0$ such that

$$|\eta, \chi|_W \leq \frac{1}{\gamma_s} \sup_{(\vec{v}, \psi) \in V_h} \frac{B(\vec{v}, \psi; \eta, \chi)}{\|\vec{v}, \psi\|_V} + \hat{C} \epsilon \|\eta, \chi\|_{L^2}, \quad \forall (\eta, \chi) \in W_h. \quad (5.33)$$

The following theorem is a special case of Theorem 1.

Theorem 2 *Let $(\vec{u}, \theta, \phi, \lambda)$ be the solution of (3.14). Suppose $h^2 < \tilde{C}\epsilon$ for some constant \tilde{C} . Then there is a constant \bar{C} , depending only on $\gamma_s, \hat{C}, \tilde{C}$, and the coercivity and continuity constants of the bilinear forms A and C defined in Section 3, such that*

$$\begin{aligned} & \|\vec{u} - \vec{u}_h, \theta - \theta_h\|_V + |\phi - \phi_h, \lambda - \lambda_h|_W + \epsilon \|\phi - \phi_h, \lambda - \lambda_h\|_{L^2} \\ & \leq \bar{C} \inf_{\substack{(\eta, \chi) \in W_h \\ (\vec{v}, \psi) \in V_h}} (\|\vec{u} - \vec{v}, \theta - \psi\|_V + |\phi - \eta, \lambda - \chi|_W + \epsilon \|\phi - \eta, \lambda - \chi\|_{L^2}), \end{aligned} \quad (5.34)$$

where $(\vec{u}_h, \theta_h) \in V_h$ and $(\phi_h, \lambda_h) \in W_h$ is the unique solution of

$$\begin{aligned} A(\vec{u}_h, \theta_h; \vec{v}, \psi) + B(\vec{v}, \psi; \phi_h, \lambda_h) &= F(\vec{v}), \quad \forall (\vec{v}, \psi) \in V_h, \\ B(\vec{u}_h, \theta_h; \eta, \chi) - \epsilon^2 C(\phi_h, \lambda_h; \eta, \chi) &= 0, \quad \forall (\eta, \chi) \in W_h. \end{aligned} \quad (5.35)$$

6 Relaxing the condition “ $h^2 < C\epsilon$ ”

Strictly speaking, the condition $h^2 < C\epsilon$ means that the mesh size depends on the thickness of the shell. Although we feel that in practice $h^2 < C\epsilon$ is not very restrictive, we have two ways to relax this condition.

First, one can stabilize the abstract problem (4.17) by replacing ϵ^2 by $\epsilon^2 + h^4$, and consider the following problem:

Perturbed Problem $V - W$: Find $U^h \in V$ and $P^h \in W$ such that

$$\begin{aligned}
A(U^h, Z) + B(Z, P^h) &= F(Z), & \forall Z \in V, \\
B(U^h, Q) - (\epsilon^2 + h^4)C(P^h, Q) &= 0, & \forall Q \in W.
\end{aligned} \tag{6.36}$$

This technique is widely used for Reissner-Mindlin plate models (e.g. [13]); in most of the cases ϵ^2 is replaced by $\epsilon^2 + h^2$. Subtracting (6.36) from (4.17),

$$\begin{aligned}
A(U - U^h, Z) + B(Z, P - P^h) &= 0, & \forall Z \in V, \\
B(U - U^h, Q) - (\epsilon^2 + h^4)C(P - P^h, Q) &= -h^4C(P, Q), & \forall Q \in W.
\end{aligned}$$

Replace Z by $U - U^h$ and Q by $P - P^h$, then subtract the second equation from the first,

$$A(U - U^h, U - U^h) + (\epsilon^2 + h^4)C(P - P^h, P - P^h) = h^4C(P, P - P^h).$$

This means

$$\|U - U^h\|_V + (\epsilon + h^2)\|P - P^h\|_W \leq Ch^2\|P\|_W,$$

and it follows that

$$\|P - P^h\|_W \leq C\|U - U^h\|_V \leq Ch^2\|P\|_W.$$

If $\|P\|_W$ is bounded by a norm of F with a coefficient independent of ϵ , then we would have uniform convergence. This is indeed the case for Reissner-Mindlin plate with clamped boundary conditions, where the L^2 norm of the shear stress is bounded by the L^2 norm of the load. For shells, such a result is not known. For the Reissner-Mindlin plate with simply supported boundary conditions, it is known[2] that the L^2 norm of the shear stress is not uniformly bounded when ϵ tends to zero. Therefore this stabilization approach does not seem to be very promising for the shell problem with simply supported boundary conditions.

However, we have the next method, which requires us to use larger bubble spaces, but will definitely satisfy the stability assumption **A2**.

Recall $\pi_h : H_{\Gamma_d}^1 \rightarrow M_{k+2}^{k+3}(\mathcal{T}_h)$. Here we redefine this operator, but still call it π_h .

Lemma 3 *There is an operator $\pi_h : H_{\Gamma_d}^1 \rightarrow M_{k+2}^{k+m}$, for $m \geq 3$, such that (5.25), (5.26), (5.28) and (5.29) hold for any $p \in L_k^0(\mathcal{T}_h)$, and in addition, (5.27) holds for any $p \in L_{k+m-3}^0(\mathcal{T}_h)$.*

Most part of the proof of this lemma is the same as in the proof of Lemma 7 of [1]; just the bubble adjustment is different. But we carry out the whole

proof here for the benefit of the reader.

Proof:

First, let $\pi_h^0 : H_{\Gamma_d}^1 \rightarrow L_1^1(\mathcal{T}_h) \cap H_{\Gamma_d}^1$ satisfy

$$\|v - \pi_h^0 v\|_{0,\tau} + h_\tau \|v - \pi_h^0 v\|_{1,\tau} \leq Ch_\tau \|v\|_{1,\tilde{\tau}}, \quad \text{for all } \tau \in \mathcal{T}_h, \quad v \in H_{\Gamma_d}^1, \quad (6.37)$$

where h_τ is the diameter of the triangle τ in \mathcal{T}_h , $\tilde{\tau}$ is the union of all the triangles in \mathcal{T}_h which meet τ [14].

Second, let $\pi_h^1 : H_{\Gamma_d}^1 \rightarrow L_{k+2}^1(\mathcal{T}_h) \cap H_{\Gamma_d}^1$ satisfy $\pi_h^1 v = 0$ at the vertices of all the triangles in \mathcal{T}_h and

$$\begin{aligned} \int_e (v - \pi_h^1 v) p ds &= 0, \quad \text{for all } e \in \mathcal{E}_h, \quad v \in H_{\Gamma_d}^1, \quad p \in \mathcal{P}_k, \\ \int_\tau (v - \pi_h^1 v) p dx &= 0, \quad \text{for all } \tau \in \mathcal{T}_h, \quad v \in H_{\Gamma_d}^1, \quad p \in \mathcal{P}_{k-1}. \end{aligned} \quad (6.38)$$

Here \mathcal{E}_h is the set of all the edges of all the triangles in \mathcal{T}_h . By a scaling argument, one can show that

$$\|v - \pi_h^1 v\|_{0,\tau} + h_\tau \|v - \pi_h^1 v\|_{1,\tau} \leq C(\|v\|_{0,\tau} + h_\tau \|v\|_{1,\tau}) \quad (6.39)$$

for all $\tau \in \mathcal{T}_h$ and $v \in H_{\Gamma_d}^1$.

Third, let $\pi_h^2 : H_{\Gamma_d}^1 \rightarrow B_{k+m}(\mathcal{T}_h)$, $m \geq 3$, satisfy

$$\int_\tau (v - \pi_h^2 v) p dx = 0, \quad \text{for all } \tau \in \mathcal{T}_h, \quad v \in H_{\Gamma_d}^1, \quad p \in \mathcal{P}_{k+m-3}. \quad (6.40)$$

A scaling argument on this operator gives

$$\|\pi_h^2 v\|_{0,\tau} + h_\tau \|\pi_h^2 v\|_{1,\tau} \leq C\|v\|_{0,\tau}, \quad \text{for all } \tau \in \mathcal{T}_h, \quad v \in H_{\Gamma_d}^1. \quad (6.41)$$

Finally, by setting

$$\pi_h v = \pi_h^0 v + \pi_h^1(v - \pi_h^0 v) + \pi_h^2(v - \pi_h^0 v - \pi_h^1(v - \pi_h^0 v)), \quad (6.42)$$

and straightforward computation, one can verify that (5.25), (5.26), (5.28) and (5.29) hold for any $p \in L_k^0(\mathcal{T}_h)$, and (5.27) holds for any $p \in L_{k+m-3}^0(\mathcal{T}_h)$. \square

Now, if we use the definition of Π_h in Lemma 2 with π_h replaced by the newly defined π_h , and assume that the geometric coefficients $b_{\alpha\beta}$, b_β^δ and $\Gamma_{\alpha\beta}^\delta$ are in

$C^{m-3,1}$ on each element, then we can uniformly approximate these coefficients by piecewise polynomials of degree $m - 3$, and prove that

$$\begin{aligned} |\eta, \chi|_{W_{\approx}} &= \sup_{(\vec{v}, \psi) \in V_{\approx}} \frac{B(\vec{v}, \psi; \eta, \chi)}{\|\vec{v}, \psi\|_V} \\ &\leq C \sup_{(\vec{v}, \psi) \in V_h} \frac{B(\vec{v}, \psi; \eta, \chi)}{\|\vec{v}, \psi\|_V} + Ch^{m-1} \|\eta, \chi\|_{L^2}, \end{aligned} \quad (6.43)$$

for

$$V_h = V_h^m = \{(\vec{v}, \psi) \in V \mid v_i \in M_{k+2}^{k+m}(\mathcal{T}_h), \psi_\alpha \in M_{k+1}^{k+3}(\mathcal{T}_h)\}.$$

Once $h^{m-1} < C\epsilon$, the stability assumption **A2** is satisfied, and the rest of the analysis in the previous section goes through. Therefore, for any fixed $\epsilon > 0$ and $0 < h < 1$, there is an integer m such that $h^{m-1} < \epsilon$ and (5.35) with $V_h = V_h^m$ provides an approximation to the exact solution of (3.14) with quasi-optimal error estimate independent of ϵ .

Remark: We have to recognize that the method given in this paper is only proved to be practically locking-free in the sense that in most cases $h^2 < Ct$ or $h^3 < Ct$ will be easily satisfied by a mesh size h determined by a reasonable accuracy requirement.

7 Reduced Integration Technique

The idea of reduced integration is widely used for problems which exhibit locking. Instead of specializing this idea to our method, we first present a brief and abstract description of the technique.

Consider a problem with locking: Find $U \in V$ such that

$$A(U, Z) + \epsilon^{-2}C(\mathcal{L}U, \mathcal{L}Z) = F(Z), \quad \forall Z \in V.$$

Here V is a Hilbert space, $\mathcal{L} : V \rightarrow W$ is a bounded linear operator, W is also a Hilbert space, the symmetric bilinear forms A and C are continuous and coercive on V and W respectively. From these assumptions, it is obvious that there are constants $C_1 > C_0 > 0$ such that

$$C_0 \|Z\|_V^2 \leq A(Z, Z) + \epsilon^{-2}C(\mathcal{L}Z, \mathcal{L}Z) \leq C_1 \epsilon^{-2} \|Z\|_V^2.$$

But if one directly discretize the problem, the error estimate will have the coefficient $\epsilon^{-2}C_1/C_0$, which is a typical locking phenomenon. The usual way of “unlocking” is the following locking-free reformulation:

Let $P = \epsilon^{-2}\mathcal{L}U$ and $B(Z, Q) = C(\mathcal{L}Z, Q)$. Then we have an equivalent saddle point problem: Find $U \in V$ and $P \in W$ such that

$$A(U, Z) + B(Z, P) = F(Z), \quad \forall Z \in V,$$

$$B(U, Q) - \epsilon^2 C(P, Q) = 0, \quad \forall Q \in W.$$

For arbitrarily chosen subspaces $V_h \subset V$ and $W_h \subset W$, one can consider the approximate solution $U_h \in V_h$ and $P_h \in W_h$ satisfying

$$A(U_h, Z) + B(Z, P_h) = F(Z), \quad \forall Z \in V_h,$$

$$B(U_h, Q) - \epsilon^2 C(P_h, Q) = 0, \quad \forall Q \in W_h.$$

If the pair (V_h, W_h) satisfies the stability condition **A1** given in Section 4, Theorem 1 of the same section holds.

The idea of reduced integration is that the unknown quantity P_h can be eliminated. One can always do this in the following way.

Define a projection operator $\mathcal{S}_h : W \rightarrow W_h$ by

$$C(\mathcal{S}_h R, Q) = C(R, Q), \quad \forall Q \in W_h.$$

Then we have $P_h = \epsilon^{-2}\mathcal{S}_h\mathcal{L}U_h$ and $B(Z, P_h) = C(\mathcal{S}_h\mathcal{L}Z, P_h)$. Hence the problem reduces to

$$A(U_h, Z) + \epsilon^{-2}C(\mathcal{S}_h\mathcal{L}U_h, \mathcal{S}_h\mathcal{L}Z) = F(Z), \quad \forall Z \in V_h.$$

When W_h is a finite element subspace consisting of discontinuous piecewise polynomials, \mathcal{S}_h is easily computable.

The advantage of the last equation above is that it involves only the primitive unknowns, however, the algebraic system is poorly conditioned. In addition to the spectrum of the operator \mathcal{L} and the elliptic operator associated to the bilinear form A , the condition number also depends on ϵ . On the other hand, the locking-free saddle point problem is much easier to solve since preconditioning and iterative techniques for such systems are better understood. In [5], for example, the positive definite systems, resulting from the reduced integration method for the Reissner-Mindlin plate model, are preconditioned by introducing mixed reformulations.

To convert our mixed method in this paper to a reduced integration method, we just need to define

$$\mathcal{L}(\vec{u}, \theta) = \left(\left(\frac{E}{2(1+\nu)} a^{\alpha\beta} \Phi_\alpha(\vec{u}, \theta) \sqrt{a} \right), \left(a^{\alpha\beta\gamma\delta} \Lambda_{\alpha\beta}(\vec{u}) \sqrt{a} \right) \right)$$

and

$$C(\underset{\sim}{\phi}, \underset{\sim}{\lambda}; \underset{\sim}{\eta}, \underset{\sim}{\chi}) = \int_{\Omega} \frac{2(1+\nu)}{E} a_{\alpha\beta} \phi^{\alpha} \eta^{\beta} \frac{1}{\sqrt{a}} dx + \int_{\Omega} a_{\alpha\beta\gamma\delta} \lambda^{\alpha\beta} \chi^{\gamma\delta} \frac{1}{\sqrt{a}} dx,$$

and use the framework given in this section.

We complete this paper with the following remarks:

Remark: Although we only described triangle elements, we believe that the technique of Lemma 1 can be used for arbitrary quadrilateral elements.

Remark: This method is designed for bending dominated shell problems; further research is needed before we will be able to say whether or not it can be used on membrane dominated shells.

Remark: The *hp* method is widely believed to be the most promising method for shells. The reasons are clear: The flexibility of high order polynomials can reduce the locking effect, and there is no deterioration of the error estimate due to the reduced integration technique. But rigorous mathematical analysis is needed to support these ideas. In this paper, our goal has been to introduce a method with lowest order polynomials for bending dominated shell problems. Membrane dominated shells, and partially bending - partially membrane dominated shells, are not considered here.

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