Enumerative Combinatorics with Fillings of Polyominoes

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- **Enumerative Combinatorics:** counting the number of elements of a finite set $S$, e.g., permutations, words, matchings, set partitions, integer partitions, paths, graphs …

- **Combinatorial statistics** $f: S \rightarrow \mathbb{N}$
  
  e.g. cycle, length, block, degree…
Outline

1. Symmetry of the longest chains
   ◦ Subsequences in permutations and words
   ◦ Crossings and nestings in matchings and graphs
   ◦ A new model: fillings of moon polyominoes

2. Combinatorics of Fillings of Moon polyominoes
   ◦ Northeast and southeast chains
   ◦ Forbidden patterns
   ◦ Transformations
   ◦ Connections to other objects
Part I: Symmetry of the longest chains

- Permutations:
  
  $732816549$ (increasing subsequence)
  $732816549$ (decreasing subsequence)

  \[
  \text{is}(w) = | \text{longest i. s.}| = 3 \\
  \text{ds}(w) = | \text{longest d. s.}| = 4
  \]

- [Deift, Baik & Johansson’99] Asymptotic distribution of $\text{is}(w)$ and $\text{ds}(w)$.

- $\text{is}(w)$ and $\text{ds}(w)$ are symmetric.
Crossings and nestings in matchings of $[2n]$ are symmetric!

\[
\begin{array}{cc}
\text{cr}_2 & \text{ne}_2 \\
2 & 0 \\
1 & 1 \\
1 & 1 \\
0 & 2 \\
\end{array}
\]

# noncrossing matchings of $[2n]$ = # nonnesting matchings of $[2n]$
k-crossings/nestings

**Theorem** [Chen, Deng, Du]

# matchings of [2n] with no 3-crossings

= # matching of [2n] with no 3-nestings

= # pairs of noncrossing Dyck paths

**Conjecture:**

#Matchings of [2n] with no k-crossings

= # Matchings of [2n] with no k-nestings
Crossing and nesting number

# k-crossing and # k-nesting: not symmetric
How about “maximal crossing” and “maximal nesting”?

- For a matching $M$,
  \[ cr(M) = \max\{ k : M \text{ has a } k\text{-crossing} \} \]
  \[ ne(M) = \max\{ k : M \text{ has a } k\text{-nesting} \} \]

Goal: symmetry between $cr$ and $ne$
Main result on Matchings

**Theorem** [Chen, Deng, Du, Stanley & Y, 07]
The pair \((\text{cr}(M), \text{ne}(M))\) has a symmetric joint distribution over all matchings on \([2n]\).

**Corollary.**
\[
\# \text{ matchings with no } k\text{-crossing} = \# \text{ matchings with no } k\text{-nesting}
\]
Idea:

- Oscillating tableau: a sequence of Ferrers diagrams \( \emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset \) s.t.

\[
\lambda^i = \lambda^{i-1} \pm -\square
\]
Theorem [Stanley & Sundaram’90]
There is a bijection between matchings of [2n] and oscillating tableaux of length 2n.

- It is realized by using standard Young tableaux and applying the RSK algorithm.

Theorem [CDDSY]
Taking conjugation in the tableaux exchanges $cr(M)$ and $ne(M)$. 
Set Partitions of \([n]\)

- A graphical representation
  \[\pi = \{1, 4, 5, 7\} \{2, 6\} \{3\}\]

- **Theorem.** [CDDSY]
  \((\text{cr}(\pi), \text{ne}(\pi))\) has a symmetric distribution over all partitions of \([n]\).
Filling of the staircase

Crossing: anti-identity submatrix (NE-chain)
Nesting: identity submatrix (SE-chain)
An extension to Ferrers diagram

01-filling of any Ferrers diagram $F$
Every row/column has at most one 1.

NE-chain $J_k$  

SE-chain $I_k$
Ferrers diagram

NE(F) = |longest NE chain|
SE(F) = |longest SE chain|

[Krattenthaler’06]
Given a Ferrers diagram F and an integer n, then (NE(F), SE(F)) has a symmetric distribution over 01-fillings of F with n 1’s..
Generalized triangulation of n-gon

**k-triangulation:**

*no k+1 diagonals that are mutually intersecting*
Results about $k$-triangulation

- **[Capoyleas & Pach’92]**
  
  $k$-triangulations of an $n$-gon has at most $k(2n-2k-1)$ lines.

- **[Dress, Koolen & Moulton’02]**
  
  maximal $k$-triangulation always has $k(2n-2k-1)$ lines

- **[Jonsson’05]**
  
  #maximal $k$-triangulations
  
  =a determinant of Catalan numbers.
Catalan number implies symmetry!

try to avoid
[Jonsson’05, Jonsson & Welker’07]: The number of 01-fillings with $m$ nonzero entries that avoid the matrix $J_k$ depends only on the size of the columns, not on the ordering of the columns.
Moon polyominoes

[Rubey’11]: The number of 01- fillings with $m$ nonzero entries that avoid the matrix $J_k$ depends only on the size of the columns, not on the ordering of the columns.

Symmetry between $I_k$ and $J_k$: flipping the moon polyomino
The General Model: *fillings of moon polyominoes*

- **Polyomino**: a finite set of square cells

- **Moon polyomino:**
  - Convex
  - intersection-free (*no skew shape*)

![Polyomino examples](image)
Fillings of moon polyominoes

- Assign an integer to each square

- Permutations
- Words
- Matchings
- Set partitions
- Graphs
- Ferrers diagram
- Stack polyomino
- Moon polyomino
Part II: Combinatorics of fillings of moon polyominoes

- Northeast and southeast chains
- Forbidden patterns
- Transformations
- Connections to other objects
The model is general:
Example 1. Chains of length 2

Permutation: *inversion* and *coinversion*

\[ \pi = 624153 \]

- *inversion*: \{ (i - j): i > j \}
- *coinversion*: \{ (i - j): i < j \}

\[ \text{inv}(\pi) = 9 : \{ 62, 64, 61, 65, 63, 21, 41, 43, 53 \} \]

\[ \text{coinv}(\pi) = 6: \{ 24, 25, 23, 45, 15, 13 \} \]

\[ \sum_{\pi} p^{\text{inv}(\pi)} q^{\text{coinv}(\pi)} = \prod_{k=1}^{n} [k]_{p,q} \]

where \([k]_{p,q} \) is the \((p,q)\)-integer \( p^{k-1} + p^{k-2}q + \ldots + pq^{k-2} + q^{k-1} \).
On words over \{1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\}

- A word is an arrangement of $1^{n_1}, 2^{n_2}, \ldots, k^{n_k}$

$$\sum_{W} p^{\text{inv}(w)} q^{\text{coinv}(w)} = \binom{n}{n_1, \ldots, n_k}_{p,q}.$$ 

- Similar results for
  - Matchings [de Sainte-Catherine’83]
  - Set partitions [Kasraoui & Zeng’06]
  - Linked partitions [Chen, Wu & Y’08]
  - Crossing and alignment for permutations [Corteel’07]
Theorem [Kasraoui’10]

The pair \((\text{ne}_2, \text{se}_2)\) has a symmetric joint distribution over the set of 01-fillings of a moon polyomino with any given column sum.
Four mixed statistics

- Bicolor the rows of $M$: $(S, M-S)$

**top-mixed statistic**

$\alpha(S, M)$:

$$
\begin{array}{c}
1 \\
1 \end{array}
\quad \text{and} \quad
\begin{array}{c}
1 \\
1 \end{array}
$$

**bottom-mixed statistic**

$\beta(S, M)$:

$$
\begin{array}{c}
1 \\
1 \end{array}
\quad \text{and} \quad
\begin{array}{c}
1 \\
1 \end{array}
$$
Four mixed statistics

- Bicolor the columns of $M$: $(T, M-T)$

left-mixed statistic $\gamma(T,M)$:

right-mixed statistic $\delta(T,M)$:
Symmetry on mixed statistics

**Theorem.** [Chen, Wang, Y, Zhao’10]
Let $\lambda(A,M)$ be any of these four mixed statistics. Then the joint distribution of the pair

$$(\lambda(A, M), \lambda(M-A, M))$$

is symmetric and independent of the subset $A$.

Note:

$$(\lambda(\emptyset, M), \lambda(M, M)) = (se2(M), ne2(M))$$
$$(\lambda(M, M), \lambda(\emptyset, M)) = (ne2(M), se2(M))$$

Special case for permutations: Chebikin’08.
The model is special enough!

Many things happen inside rectangles!
Example 2: $k$-noncrossing vs $k$-nonnesting

Problem: \# fillings with no $k$-crossing = \# fillings with no $k$-nesting

Method: Start with a filling with no $k$-crossing, then replace every appearance of $k$-nesting by other patterns.

- [Backelin, West, Xin’07] for 01-fillings of Ferrers diagrams
- [de Mier’07] for multi-graphs with fixed degree sequences
It applies to other patterns.

- Both papers compared patterns $J_k$ and $J_{k-1}$.

$$F_k = \begin{bmatrix} J_{k-1} & 0 \\ 0 & 1 \end{bmatrix}$$

- One can get more Wilf-equivalent pairs.

$$\begin{bmatrix} C & 0 \\ 0 & A \end{bmatrix}, \quad \begin{bmatrix} D & 0 \\ 0 & A \end{bmatrix}$$
Example 3. The major index

- For a word $a_1 \ a_2 \ldots \ a_n$, a descent is a position $i$ such that $a_i > a_{i+1}$.
- $\text{maj}(w) = \sum \{ i : \ i \in \text{DES}(w) \}$.
- [MacMohan’1916] The major index is equadistributed to $\text{inv}(w)$ over words.
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[Chen, Poznanovik, Y & Yang’10]
The major index can be extended to 01-fillings of moon polyominoes, which has the same distribution as $\text{ne}_2$. 
Foata’s map $\Phi$ with $\text{inv}(\Phi(w)) = \text{maj}(w)$

- **Recursive Definition:**
  - If $w$ has length 1, $\Phi(w) = w$.
  - Otherwise, $w = w' a$, then
    \[ \Phi(w) = \gamma_a(\Phi(w')) a \]

\[ w = w_1 \cdots w_{n-1} \ a \]

\[ \Phi \]

\[ \Gamma_a \]

\[ v_1 \cdots v_{n-1} \ a \]

\[ v_1 \cdots v_{n-1} \]

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Many transformations!

- [CPYY] Foata-type transformations can be defined on fillings of left-aligned stack polyominoes which carry $\text{maj}$ to $\text{ne}_2$
From polyomino to polyomino

- Bijection $f$ from fillings of $M$ to fillings of $N$ s.t. $\text{maj}(F) = \text{maj}(f(F))$
- Bejection $g$ from fillings of $M$ to fillings of $N$ s.t. $\text{ne}_2(F) = \text{ne}_2(g(F))$
And more…

- Lattice path counting and descents in Ferrers diagrams
- Integer sequences
- Pattern avoidance and appearances
- Poset, P-partitions …
Relation to other areas...

- Free probability — noncrossing diagrams
- Graph optimization and layout

- Combinatorial computational biology
RNA pseudo knot structures
Thank you very much!