Enumerative Combinatorics with Fillings of Polyominoes

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• Enumerative Combinatorics: counting the number of elements of a finite set $S$, e.g., permutations, words, matchings, set partitions, integer partitions, paths, graphs …

• Combinatorial statistics $f: S \rightarrow \mathbb{N}$
  e.g. cycle, length, block, degree...
Outline

1. Symmetry of the longest chains
   ◦ Subsequences in permutations and words
   ◦ Crossings and nestings in matchings and graphs
   ◦ A new model: fillings of moon polyominoes

2. Combinatorics of Fillings of Moon polyominoes
   ◦ Northeast and southeast chains
   ◦ Forbidden patterns
   ◦ Transformations
   ◦ Connections to other objects
Part I: Symmetry of the longest chains

- **Permutations:**
  - $732816549$ (increasing subsequence)
  - $732816549$ (decreasing subsequence)
  
  \[
  \text{is}(w) = |\text{longest i.s.}| = 3
  \]
  \[
  \text{ds}(w) = |\text{longest d.s.}| = 4
  \]

- [Deift, Baik & Johansson’99] Asymptotic distribution of $\text{is}(w)$ and $\text{ds}(w)$.

- $\text{is}(w)$ and $\text{ds}(w)$ are symmetric.
Crossings and nestings in matchings of \([2n]\) are symmetric!

\[(cr_2, ne_2)\] are symmetric!

e.g. \[
\begin{array}{cc}
2 & 0 \\
1 & 1 \\
1 & 1 \\
0 & 2 \\
\end{array}
\]

\# noncrossing matchings of \([2n]\)

= \# nonnesting matchings of \([2n]\)

= nth Catalan number
**Conjecture:**

\[
\#\text{Matchings of } [2n]\text{ with no } k\text{-crossings} = \#\text{ Matchings of } [2n]\text{ with no } k\text{-nestings}
\]
Crossing and nesting number

For a matching $M$, 
\[
\text{cr}(M) = \max \{ k : M \text{ has a } k\text{-crossing} \} \\
\text{ne}(M) = \max \{ k : M \text{ has a } k\text{-nesting} \}
\]

What’s the relation between \text{cr} and \text{ne}?
Main result on Matchings

**Theorem** [Chen, Deng, Du, Stanley & Y, 07]
The pair \((\text{cr}(M), \text{ne}(M))\) has a symmetric joint distribution over all matchings on \([2n]\).

**Corollary.**
\[
\# \text{ matchings with no k-crossing} = \# \text{ matchings with no k-nesting}
\]
Idea:

- **Oscillating tableau**: a sequence of Ferrers diagrams $\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset$ s.t. 

$$\lambda^i = \lambda^{i-1} \pm \square$$
Theorem [Stanley & Sundaram’90]
There is a bijection between matchings of 
[2n] and oscillating tableaux of length 2n.

- It is realized by using standard Young 
tableaux and applying the RSK algorithm.

Theorem [CDDSY]
Taking conjugation in the tableaux 
exchanges cr(M) and ne(M).
Set Partitions of $[n]$

- A graphical representation
  \[ \pi = \{1, 4, 5, 7\} \{2, 6\} \{3\} \]

- **Theorem.** [CDDSY]
  \((cr(\pi), ne(\pi))\) has a symmetric distribution over all partitions of $[n]$. 

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Filling of the triangular board

Crossing: anti-identity submatrix (NE-chain)
Nesting: identity submatrix (SE-chain)
An extension to Ferrers diagram

01-filling of any Ferrers diagram $F$
Every row/column has at most one 1.

**NE-chain** $J_k$

**SE-chain** $I_k$

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[Krattenthaler’06]
Given a Ferrers diagram \( F \) and an integer \( n \), then \((\text{NE}(F), \text{SE}(F))\) has a symmetric distribution over \( 01 \)-fillings of \( F \) with \( n \) 1’s.
Generalized triangulation of n-gon

$k$-triangulation:
no \( k+1 \) diagonals that are mutually intersecting
Results about k-triangulation

- [Capoyleas & Pach’92] k-triangulations of an n-gon has at most $k(2n-2k-1)$ lines.
- [Dress, Koolen & Moulton’02] maximal k-triangulation always has $k(2n-2k-1)$ lines
- [Jonsson’05] #maximal k-triangulations = a determinant of Catalan numbers.
Catalan number implies symmetry!

try to avoid
[Jonsson’05, Jonsson & Welker’07]: The number of 01-fillings with $m$ nonzero entries that avoid the matrix $J_k$ depends only on the size of the columns, not on the ordering of the columns.
[Rubey’11]: The number of 01-fillings with \( m \) nonzero entries that avoid the matrix \( J_k \) depends only on the size of the columns, not on the ordering of the columns.

Symmetry between \( I_k \) and \( J_k \): flipping the moon polyomino
The General Model: **fillings of moon polyominoes**

- **Polyomino**: a finite set of square cells

- **Moon polyomino**:
  - Convex
  - intersection-free (*no skew shape*)

![Polyomino Examples](image-url)
Fillings of moon polyominoes

- Assign an integer to each square

Permutations $\rightarrow$ Words $\rightarrow$ Matchings $\rightarrow$ Set partitions

Graphs $\rightarrow$ Ferrers diagram $\rightarrow$ Stack polyomino $\rightarrow$ Moon polyomino

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Part II: Combinatorics of fillings of moon polyominoes

- Northeast and southeast chains
- Forbidden patterns
- Transformations
- Connections to other objects
The model is general: Example 1. Chains of length 2

Permutation: *inversion* and *coinversion*

π = 624153

- *inversion*: \{(i - j): i > j \}
- *coinversion*: \{(i - j): i < j \}

\[ \text{inv}(\pi) = 9 : \{62, 64, 61, 65, 63, 21, 41, 43, 53\} \]

\[ \text{coinv}(\pi) = 6: \{24, 25, 23, 45, 15, 13\} \]

\[ \sum_{\pi} p^{\text{inv}(\pi)} q^{\text{coinv}(\pi)} = \prod_{k=1}^{n} [k]_{p,q} \]

where \([k]_{p,q}\) is the \((p,q)\)-integer \(p^{k-1} + p^{k-2}q + \ldots + pq^{k-2} + q^{k-1}\).
On words over \( \{1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\} \)

- A word is an arrangement of \(1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\)

\[
\sum_{W} p^{inv(w)} q^{coinv(w)} = \binom{n}{n_1, \ldots, n_k}_{p,q}.
\]

- Similar results for
  - Matchings [de Sainte-Catherine'83]
  - Set partitions [Kasraoui & Zeng'06]
  - Linked partitions [Chen, Wu & Y' 08]
  - Crossing and alignment for permutations [Corteel'07]
Theorem [Kasraoui’10]

The pair \((\text{ne}2, \text{se}2)\) has a symmetric joint distribution over the set of 01-fillings of a moon polyomino with any given column sum.
Four mixed statistics

- Bicolor the rows of $M$: $(S, M-S)$

**top-mixed statistic** $\alpha(S,M)$:

**bottom-mixed statistic** $\beta(S,M)$:
Four mixed statistics

- Bicolor the columns of $M$: $(T, M-T)$

**left-mixed statistic** $\gamma(T, M)$:

**right-mixed statistic** $\delta(T, M)$:

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Symmetry on mixed statistics

Theorem. [Chen, Wang, Y, Zhao’10]
Let $\lambda(A,M)$ be any of these four mixed statistics. Then the joint distribution of the pair

$$(\lambda(A, M), \lambda(M-A, M))$$

is symmetric and independent of the subset $A$.

Note:

$$(\lambda(\emptyset, M), \lambda(M, M)) = (\text{se}2(M), \text{ne}2(M))$$
$$(\lambda(M, M), \lambda(\emptyset, M)) = (\text{ne}2(M), \text{se}2(M))$$

Special case for permutations: Chebikin’08.
The model is special enough!

Many things happen inside rectangles!
Example 2: k-noncrossing vs k-nonnesting

Problem: # fillings with no k-crossing

= # fillings with no k-nesting

Method: Start with a filling with no k-crossing, then replace every appearance of k-nesting by other patterns.

- [Backelin, West, Xin’07] for 01-fillings of Ferrers diagrams
- [de Mier’07] for multi-graphs with fixed degree sequences
It applies to other patterns.

- Both papers compared patterns $J_k$ and

$$F_k = \begin{bmatrix} J_{k-1} & 0 \\ 0 & 1 \end{bmatrix}$$

- One can get more Wilf-equivalent pairs.

$$\begin{bmatrix} C & 0 \\ 0 & A \end{bmatrix}, \quad \begin{bmatrix} D & 0 \\ 0 & A \end{bmatrix}$$
Example 3. The major index

- For a word $a_1 \ a_2 \ \ldots \ a_n$, a descent is a position $i$ such that $a_i > a_{i+1}$.
- $\text{maj}(w) = \sum \{ i : i \in \text{DES}(w) \}$.
- [MacMahon'1916] The major index is equadistributed to $\text{inv}(w)$ over words.
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[Chen, Poznanovik, Y & Yang’10]

The major index can be extended to 01-fillings of moon polyominoes, which has the same distribution as $\text{ne}_2$. 
Foata’s map $\Phi$ with $\text{inv}(\Phi(w)) = \text{maj}(w)$

- **Recursive Definition:**
  - If $w$ has length 1, $\Phi(w) = w$.
  - Otherwise, $w = w' a$, then
    \[
    \Phi(w) = \gamma_a(\Phi(w')) a
    \]
Many transformations!

- [CPYY] Foata-type transformations can be defined on fillings of left-aligned stack polyominoes which carry $\text{maj}$ to $\text{ne}_2$.
From polyomino to polyomino

- Bijection $f$ from fillings of $M$ to fillings of $N$ s.t. $\text{maj}(F) = \text{maj}(f(F))$
- Bijection $g$ from fillings of $M$ to fillings of $N$ s.t. $\text{ne}_2(F) = \text{ne}_2(g(F))$
And more…

- Lattice path counting and descents in Ferrers diagrams
- Rook placement with restrictions
- Pattern avoidance and appearances
- Poset, P-partitions …
Relation to other areas…

- Free probability — noncrossing diagrams

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Graph optimization and layout: stack, queue, and arch
• Combinatorial computational biology: RNA pseudo knot structures
Thank you very much!

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