Enumerative Combinatorics with Fillings of Polyominoes

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• Enumerative Combinatorics: counting the number of elements of a finite set $S$, e.g., permutations, words, matchings, set partitions, integer partitions, paths, graphs …

• Combinatorial statistics $f: S \rightarrow N$
  e.g. cycle, length, block, degree...
Outline

1. Symmetry of the longest chains
   ◦ Subsequences in permutations and words
   ◦ Crossings and nestings in matchings and graphs
   ◦ A new model: fillings of moon polyominoes

2. Combinatorics of Fillings of Moon polyominoes
   ◦ Northeast and southeast chains
   ◦ Forbidden patterns
   ◦ Transformations
   ◦ Connections to other objects
Part I: Symmetry of the longest chains

- Permutations:
  
  732816549  (increasing subsequence)
  732816549  (decreasing subsequence)

  $\text{is}(w) = |\text{longest i.s.}| = 3$
  $\text{ds}(w) = |\text{longest d.s.}| = 4$

- [Deift, Baik & Johansson’99] Asymptotic distribution of $\text{is}(w)$ and $\text{ds}(w)$.

- $\text{is}(w)$ and $\text{ds}(w)$ are symmetric.
Crossings and nestings in matchings of \([2n]\)

- \((cr_2, ne_2)\) are symmetric!

  e.g. \[
  \begin{array}{cc}
  cr_2 & ne_2 \\
  2 & 0 \\
  1 & 1 \\
  1 & 1 \\
  0 & 2 \\
  \end{array}
  \]

  \# noncrossing matchings of \([2n]\)
  = \# nonnesting matchings of \([2n]\)
  = nth Catalan number

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**Conjecture:**

# Matchings of [2n] with no k-crossings

= # Matchings of [2n] with no k-nestings
Crossing and nesting number

# k-crossing and # k-nesting: not symmetric
How about “maximal crossing” and “maximal nesting”? 

- For a matching $M$, 
  \[ cr(M) = \max \{ k : M \text{ has a k-crossing} \} \]
  \[ ne(M) = \max \{ k : M \text{ has a k-nesting} \} \]

Goal: symmetry between $cr$ and $ne$
Main result on Matchings

**Theorem** [Chen, Deng, Du, Stanley & Y, 07]
The pair \((\text{cr}(M), \text{ne}(M))\) has a symmetric joint distribution over all matchings on \([2n]\).

\[
\#\{M: \text{cr}(M)=i, \text{ne}(M)=j\} = \#\{M: \text{cr}(M)=j, \text{ne}(M)=i\}
\]

**Corollary.**

\[
\# \text{ matchings with no k-crossing} = \# \text{ matchings with no k-nesting}
\]
Idea:

• Oscillating tableau: a sequence of Ferrers diagrams \( \emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset \) s.t.

\[ \lambda^i = \lambda^{i-1} +/ - \]

\[
\begin{array}{cccccc}
\emptyset & \square & \square & \square & \square & \emptyset \\
\emptyset & \square & \emptyset & \square & \emptyset & \emptyset \\
\end{array}
\]
Theorem [Stanley & Sundaram’90] There is a bijection between matchings of \([2n]\) and oscillating tableaux of length \(2n\).

- It is realized by using standard Young tableaux and applying the RSK algorithm.
- \((\text{cr}, \text{ne}) \iff \max \text{ row/column lengths.}\)

Theorem [CDDSY] Taking conjugation in the tableaux exchanges \(\text{cr}(M)\) and \(\text{ne}(M)\).
Set Partitions of \([n]\)

- A graphical representation
  \[\pi = \{1, 4, 5, 7\} \{2, 6\} \{3\}\]

- **Theorem.** [CDDSY]
  \((\text{cr}(\pi), \text{ne}(\pi))\) has a symmetric distribution over all partitions of \([n]\).
Filling of the triangular board

Crossing: anti-identity submatrix (NE-chain)
Nesting: identity submatrix (SE-chain)
An extension to Ferrers diagram

01-filling of any Ferrers diagram $F$

Every row/column has at most one 1.

NE-chain $J_k$

SE-chain $I_k$
Ferrers diagram

\[ \text{NE}(F) = |\text{longest NE chain}| \]
\[ \text{SE}(F) = |\text{longest SE chain}| \]

[Krattenthaler’06]
Given a Ferrers diagram \( F \) and an integer \( n \), then \((\text{NE}(F), \text{SE}(F))\) has a symmetric distribution over restricted 01-fillings of \( F \) with \( n \) 1’s.
Generalized triangulation of n-gon

k-triangulation:
no \( k+1 \) diagonals that are mutually intersecting
Results about k-triangulation

- [Capoyleas & Pach’92] k-triangulations of an n-gon has at most $k(2n-2k-1)$ lines.
- [Dress, Koolen & Moulton’02] maximal k-triangulation always has $k(2n-2k-1)$ lines.
- [Jonsson’05] #maximal k-triangulations =a determinant of Catalan numbers.
Catalan number implies symmetry!

try to avoid

\[
\begin{array}{cccc}
\star & \star & \star & \star \\
\star & & & \\
& & & \\
& & \star & \\
& \star & & \\
\end{array}
\]

\[
\begin{array}{cccc}
\star & \star & \star & \star \\
\star & & & \\
& & & \\
& & \star & \\
& \star & & \\
\end{array}
\]

\[
F(\text{max 1's, no k-NE}) = F(\text{max 1's, no k-SE})
\]
Stack polyominoes

[Jonsson’05, Jonsson & Welker’07]:
\[ F_{01}(L, n, ne < k) = F_{01}(L, n, se < k) \]
where \( n \) is the number of ones in the filling.
[Rubey’11]:
\[ F(M, n, ne<k) = F(M, n, se<k) \]
And \[ F_{01}(M, n, ne<k) = F_{01}(M, n, se<k) \]
The General Model:
fillings of moon polyominoes

- **Polyomino**: a finite set of square cells

- **Moon polyomino**:
  - Convex
  - intersection-free (no skew shape)
Fillings of moon polyominoes

- Assign an integer to each square

Permutations \(\rightarrow\) Words \(\rightarrow\) Matchings \(\rightarrow\) Set partitions

Graphs \(\rightarrow\) Ferrers diagram \(\rightarrow\) Stack polyomino \(\rightarrow\) Moon polyomino
Part II: Combinatorics of fillings of moon polyominoes

- Northeast and southeast chains
- Forbidden patterns
- Transformations
- Connections to other objects
The model is general:
Example 1. Chains of length 2

Permutation: \textit{inversion} and \textit{coinversion}

\[ \pi = 624153 \]

- \textit{inversion}: \{(i - j): i > j \}
- \textit{coinversion}: \{(i - j): i < j \}

\[ \text{inv}(\pi) = 9 : \{ 62, 64, 61, 65, 63, 21, 41, 43, 53 \} \]

\[ \text{coinv}(\pi) = 6: \{ 24, 25, 23, 45, 15, 13 \} \]

\[ \sum_{\pi} p^{\text{inv}(\pi)} q^{\text{coinv}(\pi)} = \prod_{k=1}^{n} [k]_{p,q} \]

where \([k]_{p,q}\) is the \((p,q)\)-integer \(p^{k-1} + p^{k-2}q + \ldots + pq^{k-2} + q^{k-1}\).
On words over \( \{ 1^{n_1}, 2^{n_2}, \ldots, k^{n_k} \} \)

- A word is an arrangement of \( 1^{n_1}, 2^{n_2}, \ldots, k^{n_k} \)

\[
\sum_{W} p^{\text{inv}(w)} q^{\text{coinv}(w)} = \binom{n}{n_1, \ldots, n_k}_{p,q}.
\]

- Similar results for
  - Matchings [de Sainte-Catherine'83]
  - Set partitions [Kasraoui & Zeng’06]
  - Linked partitions [Chen, Wu & Y’08]
  - Crossing and alignment for permutations [Corteel’07]
Theorem [Kasraoui’10]
The pair \((\text{ne}2, \text{se}2)\) has a symmetric joint distribution over the set of 01-fillings of a moon polyomino with any given column sum.
various mixed statistics

- Bicolor the rows of M and mix the 2-chains by the position of the top cell/bottom cell

**top-mixed statistic**

\[ \alpha(S,M): \]

**bottom-mixed statistic**

\[ \beta(S,M): \]
• Mix by the charge of a corner cell

Positive chains

Negative chains
Theorem. [Chen, Wang, Y, Zhao’10; Wang & Y’13]
Let $\lambda(A)$ be the number of any of the mixed statistics. (Hence $\lambda(M-A)$ is the number of remaining 2-chains.) Then the joint distribution of the pair $(\lambda(A), \lambda(M-A))$ is always symmetric and independent of the subset $A$.

Note: $(\lambda(\emptyset), \lambda(M)) = (\text{se}2(M), \text{ne}2(M))$
$(\lambda(M), \lambda(\emptyset)) = (\text{ne}2(M), \text{se}2(M))$

Special case for permutations: Chebikin’08.
The model is special enough!

Many things happen inside rectangles!
Example 2: k-noncrossing vs k-nonnesting

Problem: \# fillings with no k-crossing = \# fillings with no k-nesting

Method: Start with a filling with no k-crossing, then replace every appearance of k-nesting by other patterns.

- [Backelin, West, Xin’07] for 01-fillings of Ferrers diagrams
- [de Mier’07] for multi-graphs with fixed degree sequences
It applies to other patterns.

- Both papers compared patterns $J_k$ and $J_{k-1}$.

$$F_k = \begin{bmatrix} J_{k-1} & 0 \\ 0 & 1 \end{bmatrix}$$

- One can get more Wilf-equivalent pairs.

$$\begin{bmatrix} C & 0 \\ 0 & A \end{bmatrix}, \quad \begin{bmatrix} D & 0 \\ 0 & A \end{bmatrix}$$
Applies to symmetric fillings

- [Bousquet-Melou, Steingrimsson’05] symmetric 01-fillings of symmetric Ferrers diagrams – involution
- [Bloom, Saracino’12] explain the connection between algebraic and combinatorial approaches by modifying the growth diagram algorithm
The model is flexible

- One can change the fillings, or
- One can change the polyomino.
Example 3. The major index

- For a word $a_1 a_2 \ldots a_n$, a descent is a position $i$ such that $a_i > a_{i+1}$.
- $\text{maj}(w) = \sum \{ i : i \in \text{DES}(w) \}$.
- [MacMahon’1916] The major index is equadistributed to $\text{inv}(w)$ over words.
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[Chen, Poznanovik, Y & Yang’ 10] The major index can be extended to 01-fillings of moon polyominoes, which has the same distribution as $\text{ne}_2$. 
Foata’s map $\Phi$ with $\text{inv}(\Phi(w)) = \text{maj}(w)$

- **Recursive Definition:**
  - If $w$ has length 1, $\Phi(w) = w$.
  - Otherwise, $w = w' a$, then
    \[
    \Phi(w) = \gamma_a(\Phi(w')) a
    \]
Many transformations!

- [CPYY] Foata-type transformations can be defined on fillings of left-aligned stack polyominoes which carry maj to ne$_2$
From polyomino to polyomino

- Bijection $f$ from fillings of $M$ to fillings of $N$ s.t.
  $\text{maj}(F) = \text{maj}(f(F))$
- Bijection $g$ from fillings of $M$ to fillings of $N$ s.t.
  $\text{ne}_2(F) = \text{ne}_2(g(F))$
And more...

- Lattice path counting and descents in Ferrers diagrams
- Rook placement with restrictions
- Pattern avoidance and appearances
- Simplicial complexes/Schubert polynomials …
Relation to other areas…

- Free probability – noncrossing diagrams
Crossings appear in the combinatorial interpretations of

- Mixed moments of random variables
- Moments of orthogonal polynomials
- Linearization coefficients …
Graph Optimization

- **K-stack layout and k-queue layout:**
  A partition of the edges into k-sets of non-crossing (non-nesting) edges
Stack- and Queue- numbers

- Stack-number: minimum $k$ such that there is a total order of the vertices with which $G$ has a $k$-stack layout
- Queue-number: minimum $k$ such that there is a total order of the vertices with which $G$ has a $k$-queue layout
Combinatorial computational biology: RNA pseudo knot structures
THANK YOU VERY MUCH!