MATH 613. Graph Theory, Spring 2010

Assignment 6. Due on Monday, December 5, 2011

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1. Show that if the edges of K_{17} are colored with three colors, there must be a monochromatic triangle.

2. Prove the asymptotic bound for the Ramsey number $R(4,t)$: There is a constant C such that for sufficiently large t, for any $\epsilon > 0$,

$$R(4,t) > Ct^{2-\epsilon}.$$

   (Hint: Use the inequality $R(k,t) > n - \binom{n}{k}p^{\binom{k}{2}} - \binom{n}{t}(1-p)^{\binom{t}{2}}$ for $k = 4$. To simplify the calculation, take $p = x(n)n^{-1/2}$ where $x(n) \to 0$.)

3. Let S be a set containing $(n+1)m - 1$ elements, for some positive integer n. Suppose that the n-element subsets of S are partitioned into two non-empty classes.

   (a) Let $\mathcal{P}$ be a partition of n-subsets of S into two non-empty classes. Select two n-subsets A and B from different classes so that their intersection has maximal size. Prove that $|A \cap B| = n - 1$.

   (b) Let A and B are chosen as in part (a), Prove that $S - A \cup B$ is a set with $(n+1)(m-1) - 1$ elements.

   (c) Prove that there are at least m pairwise disjoint n-subsets of S that are in the same class of $\mathcal{P}$.

4. Suppose that the eigenvalues of a simple graph G are $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$, where G has n vertices and m edges. We have that $\sum \lambda_i = 0$ and $\sum \lambda_i^2 = 2m$. Give an upper bound for the largest eigenvalue λ_0. (Your answer should be better than the trivial bound $\sqrt{2m}$).

5. Let G be a simple graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. For $i = 1, 2, \ldots, n$ let G_i denote the induced subgraph on $V \setminus v_i$. Prove that

$$\frac{d(ch(G;\lambda))}{d\lambda} = \sum_{i=1}^{n} ch(G_i;\lambda).$$

6. Let G be the complete bipartite graph $K_{m,n}$. Compute the characteristic polynomial of G, and give the spectrum of G.

7. Prove that a graph G is regular and connected if and only if J is a linear combination of powers of $A(G)$, where J is the matrix in which every entry is 1.
(Hint: For necessity, note that if G is connected and k-regular, then the minimal polynomial of A can be written as $(\lambda - k)g(\lambda)$ for some polynomial $g(\lambda)$ with $g(A) \neq 0$. For regularity in the sufficiency, compare AJ and JA.)

8. Let C_n be the n-cube graph, i.e., the graph whose vertex set V is $\{0,1\}^n$, and whose edges are defined by: two vertices $u = u_1 u_2 \ldots u_n$ and $v = v_1 v_2 \ldots v_n$ are adjacent iff $u_i \neq v_i$ for exactly one $i \in [n]$. Let $\mathbb{R}[V]$ be the vector space of all the functions $f : V \to \mathbb{R}$. The space $\mathbb{R}[V]$ has a natural inner product. For $f, g \in \mathbb{R}[V]$,

$$< f, g >= \sum_{u \in \{0,1\}^n} f(u) g(u).$$

The standard basis of $\mathbb{R}[V]$ is the set $\{f_u : u \in \{0,1\}^n\}$ where $f_u(v) = \delta_{u,v}$ for $u, v \in \{0,1\}^n$. Denote by B_1 the standard basis.

(a) For any two vertices $u, v \in \{0,1\}^n$, $u \cdot v$ is defined to be $\sum_i u_i v_i$. For each $u \in \{0,1\}^n$, define a function $\chi_u \in \mathbb{R}[V]$ by letting

$$\chi_u(v) = (-1)^{u \cdot v}.$$

Prove that the set $\{\chi_u : u \in \{0,1\}^n\}$ is orthogonal with respect to the inner product of $\mathbb{R}[V]$, i.e.,

$$< \chi_u, \chi_v >= \delta_{u,v} 2^n,$$

for all $u, v \in \{0,1\}^n$.

(b) Prove that the set $\{\chi_u : u \in \{0,1\}^n\}$ forms a basis of the vector space $\mathbb{R}[V]$. Denoted by B_2 this basis.

(c) For $1 \leq i \leq n$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \{0,1\}^n$ where the only 1 occurs in position i. Let $S = \{e_1, e_2, \ldots, e_n\}$.
Define a linear transformation $\Phi : \mathbb{R}[V] \to \mathbb{R}[V]$ as follows. For $f \in \mathbb{R}[V]$, Φf is the element in $\mathbb{R}[V]$ which is given by

$$\Phi f(v) = \sum_{e_i \in S} f(v + e_i)$$

where $v + e_i$ is the usual vector addition modulo 2.
Prove that the matrix of Φ with respect to the standard basis B_1 is just $A(C_n)$, the adjacency matrix of the n-cube graph C_n.

(d) Prove that $\Phi \chi_u = \lambda_u \chi_u$ for each $u \in \{0,1\}^n$, where

$$\lambda_u = \sum_{e \in S} (-1)^{u \cdot e} = n - 2|u|,$$

where $|u|$ is the number of 1's in $u = u_1 u_2 \ldots u_n$.

(e) Compute the eigenvalues of the matrix $A(C_n)$.