Mixed Statistics on 01-Fillings of Moon Polyominoes

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inversions of a permutation

π=624153

inversions: 62, 64, 61, 65, 63, 21,41, 43, 53

coinversions: 24, 25, 23, 45, 15, 13

inv(π)= 9 coinv(π)=6

$$\boxed{\sum_{\pi} p^{inv(\pi)} q^{coinv(\pi)} = \prod_{i=1}^{n} \left(p^i + p^{i-1}q + \cdots + pq^{i-1} + q^i\right)}$$

A variation of inversion

[Chebikin] π=624153

$\iota(\pi)$ = | inversion (i, j)| +|coinversions (i, j)|

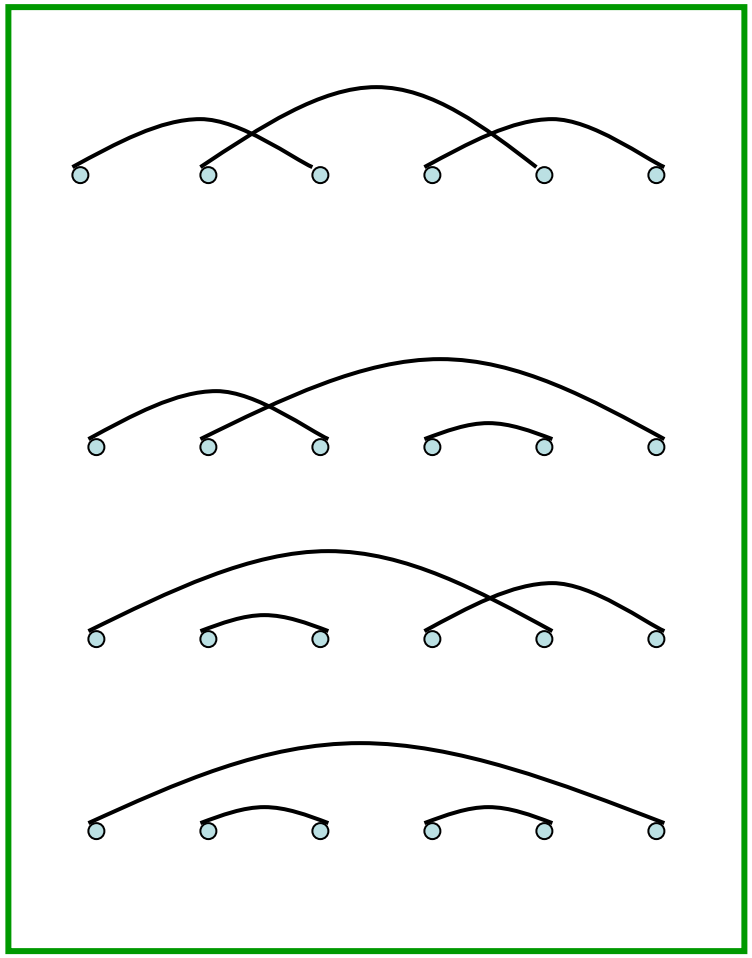
e.g. 62, 64, 61, 65, 63, 41, 43, 53,

24, 25, 23, 15, 13

$\iota(\pi)$=13

$\iota(\pi)$ has the same distribution as inv(π)

crossings and nestings in matchings

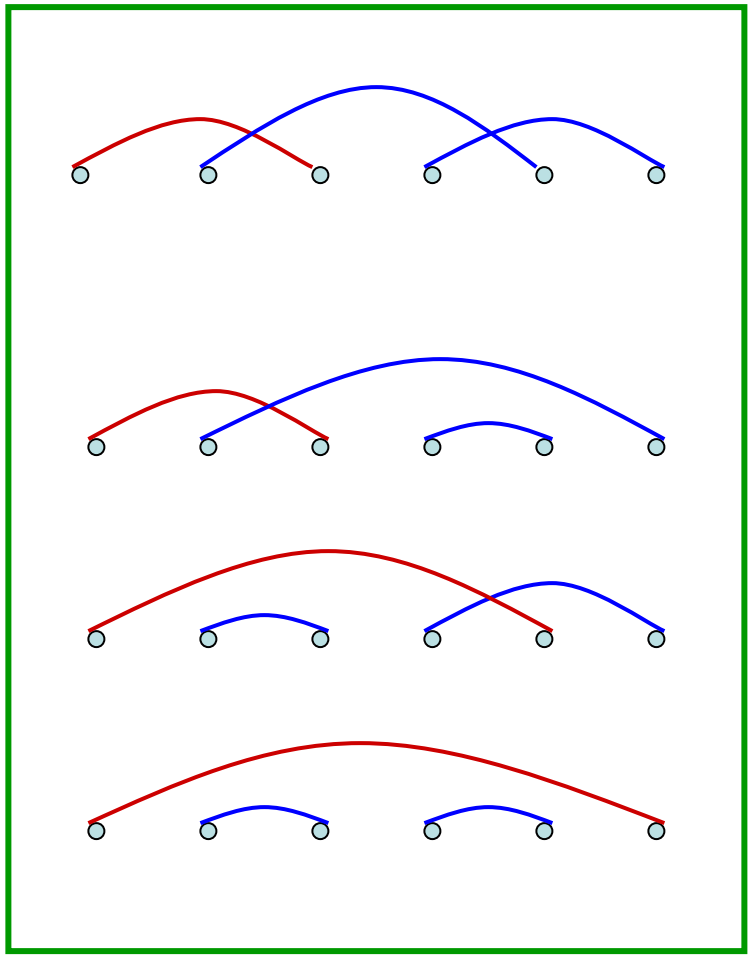


- (*cr*, *ne*) has a symmetric joint distribution.

e.g.

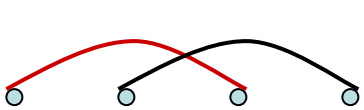
$\sum_M p^{cr(M)} q^{ne(M)} = p^2 + 2pq + q^2$

Crossings and nestings

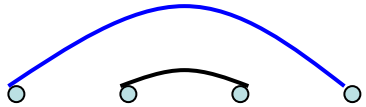


Consider a combination of *cr* and *ne*

std_1:

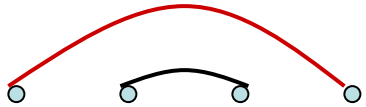


and

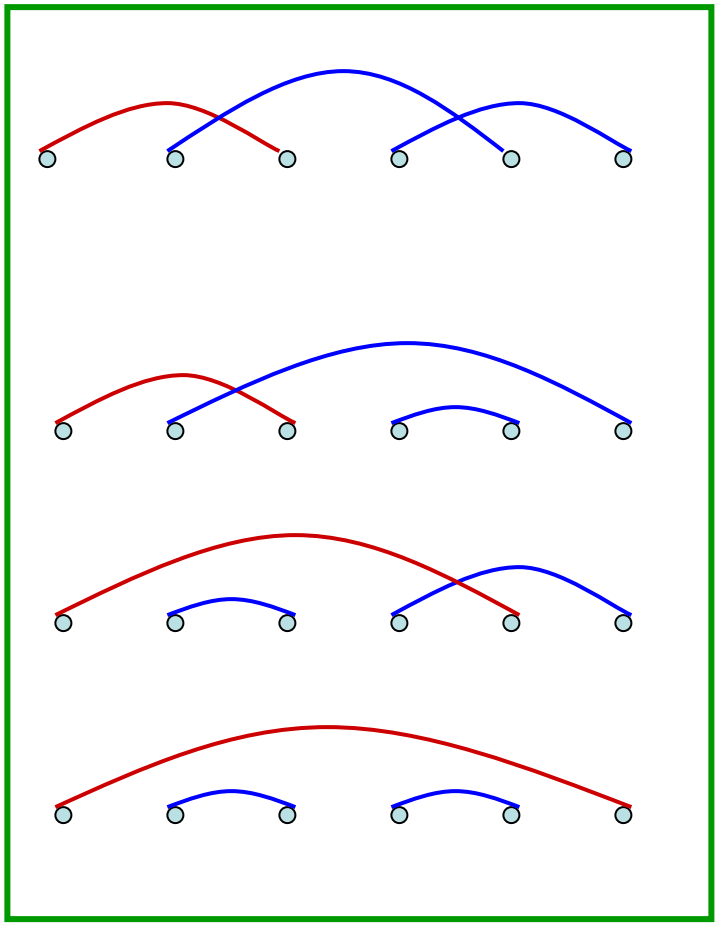


std_2:

and

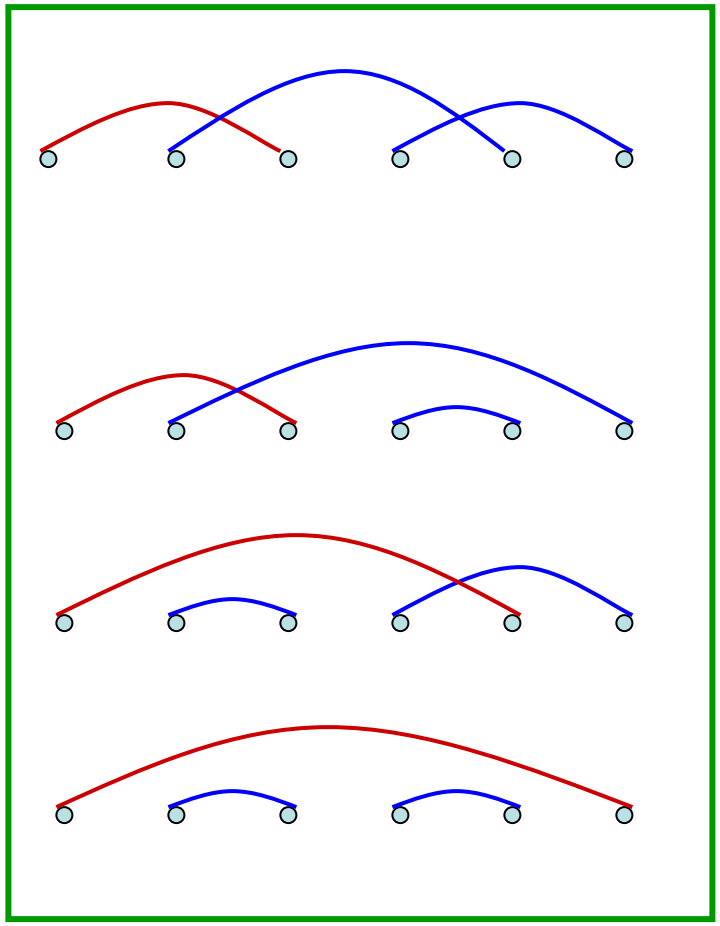


Crossings and nestings



std_1,	std_2
1	1
2	0
1	1
0	2

Crossings and nestings

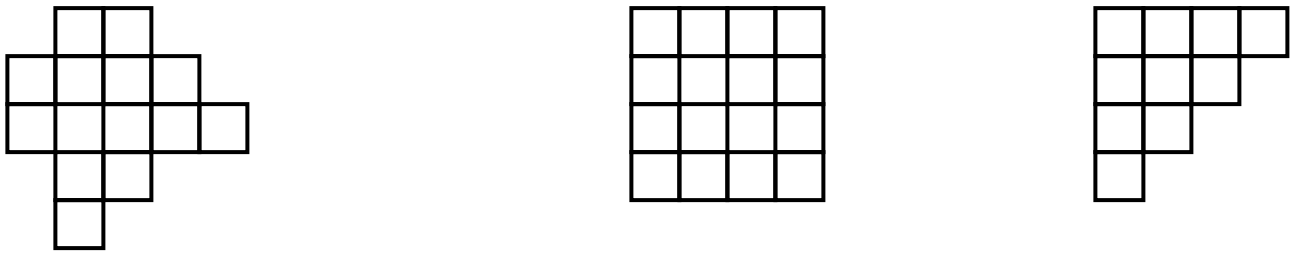


std_1,	std_2
1	1
2	0
1	1
0	2

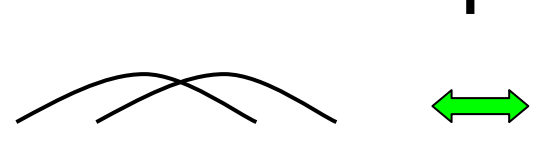
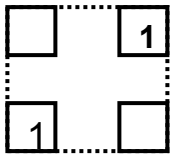
Even-odd conjecture:
(std_1, std_2) has the same symmetric distribution as (cr, ne) over all matchings with fixed left and right endpoints.

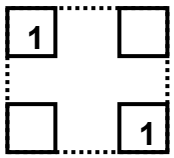
Fillings of moon polyominoes

- Convex, intersection-free polyominoes



- Assign *0* or *1* to each square

inversion ⟷  ⟷  NE-chains

coinversion ⟷  ⟷  SE-chains

Let M be a moon polyomino.

F(M,s, t) be the set of 01-fillings with

row sum $s=(s_1, \ldots, s_m) \in \{0, 1\}^n$ and

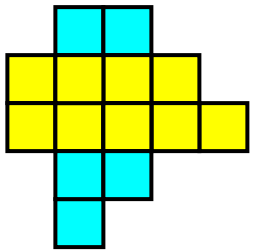
column sum $t=(t_1, \ldots, t_n) \in \mathbb{N}^n$,

Theorem. [Kasraouri]

The pair of statistics (ne, se) has a symmetric joint distribution on F(M, s, t).

Four mixed statistics

- Bicolor the rows of *M*: (S, *M*-S)

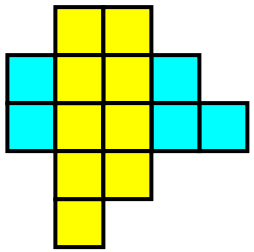


top-mixed statistic α(S,M):

bottom-mixed statistic β(S,M):

Four mixed statistics

- Bicolor the columns of *M*: (T, *M*-T)



left-mixed statistic γ(T,M):

right-mixed statistic δ(T,M):

Main Result

Theorem.

Let λ(A,M) be any of these four mixed statistics.
Then the joint distribution of the pair

(λ(A, M), λ (*M*-A, M))

is independent of the subset A.

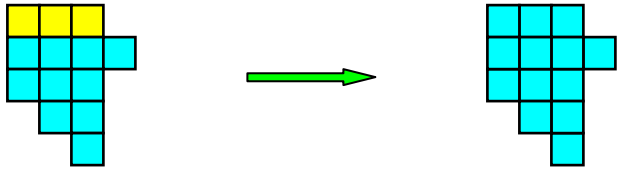
Note: ($\lambda(\emptyset$, M), λ (*M*, M)) = (*se*(M), *ne*(M))
(λ(*M*, M), $\lambda(\emptyset$, M))= (ne(M), se(M))

Cor: (λ(A, M), λ(*M*-A, M)) has a symmetric distribution.

Ideas of the proofs

- Lemma. For S={1},

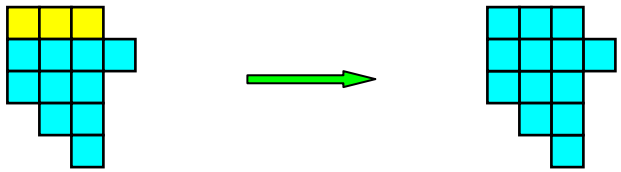
$(\alpha(S,M), \alpha(M\text{-}S, M)) \overset{d}{\sim} (se(M),ne(M))$



Idea of the proofs

- Lemma. For S={1},

$$(\alpha(S,M), \alpha(M\text{-}S, M)) \overset{d}{\sim} (se(M), ne(M))$$



- Assume $S=\{r_1, \ldots, r_s\}$ and $S'=S-\{r_s\}$. Then

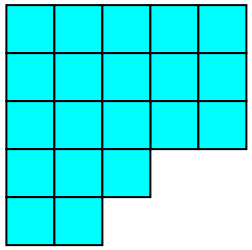
$$(\alpha(S, M), \alpha(M\text{-}S, M)) \overset{d}{\sim} (\alpha(S', M), \alpha(M\text{-}S', M))$$

Similar for the columns.

Detail for S={1}

Special case: Ferrers shape. Exactly one 1 in each row and each column

h_i: 1 2 2 2 3



Detail for S={1}

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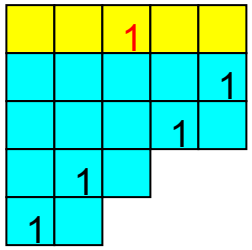
h_i: 1 2 2 2 3



Recorded by a_i: (2, 2, **1**, 1, 1)

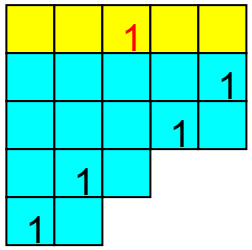
$se(M)=\sum (a_i-1)$

$ne(M)=\sum (h_i-a_i)$

Detail for S={1}

Special case: Ferrers shape. Exactly one 1 in each row and each column

h_i: 1 2 2 2 3



Recorded by a_i: (2, 2, **1**, 1, 1)

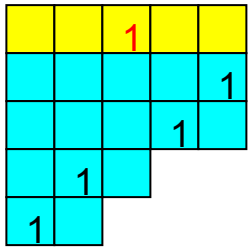
$se(M)=\sum (a_i-1)$
$ne(M)=\sum (h_i-a_i)$

Compare $\alpha(S, M)$ and se(M):
$\alpha(S, M) = se(M) - j + h_j$

Detail for S={1}

Special case: Ferrers shape. Exactly one 1 in each row and each column

h_i: 1 2 2 2 3



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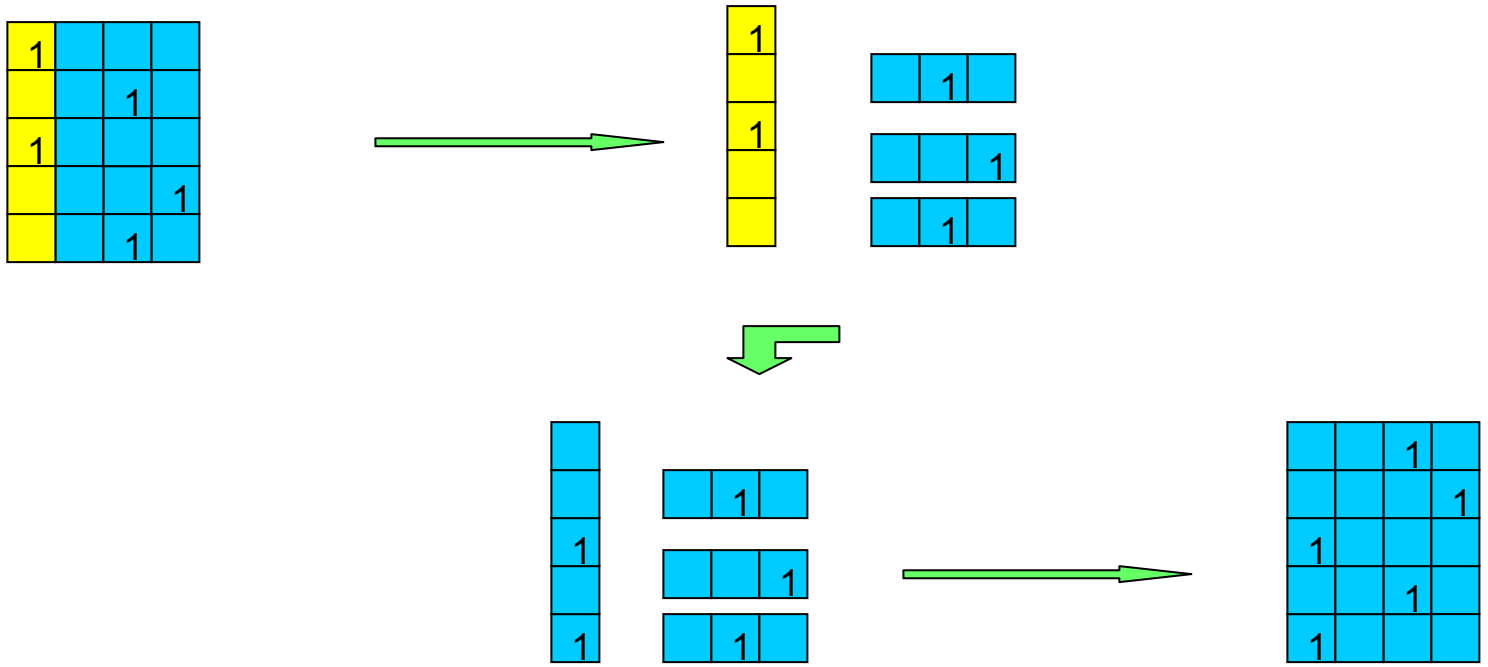
Map on fillings:

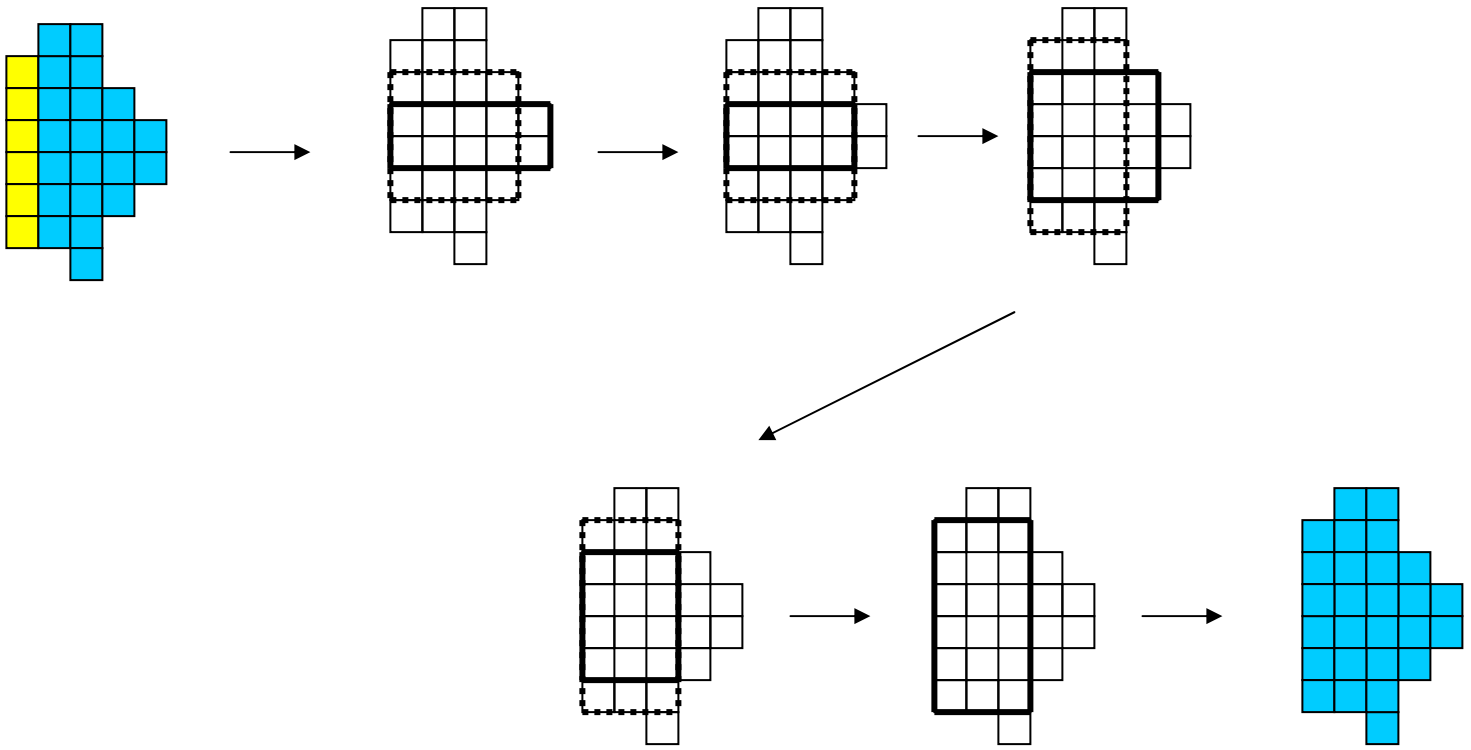
$(a_1, \ldots a_{j-1}, \mathbf{1}, a_{j+1} \ldots, a_n) \rightarrow (a_1-1, \ldots, a_{j-1}-1, \mathbf{h_j}, a_{j+1}, \ldots, a_n)$

Details for T={1}

- An involution ρ on rectangle shapes



General moon polyomino



Apply ρ iterately to change γ(T, M) to se(M).

Thank you very much.

Preprint available at

http://www.math.tamu.edu/~cyan/papers.html