Math 141 Week-in-Review 7
Sections 7.2-7.4

Key Terms:

- **relative frequency**: Suppose we repeat an experiment $n$ times and an event $E$ occurs $m$ of those times. Then $\frac{m}{n}$ is called the relative frequency of the event $E$.

- **empirical probability**: If the relative frequency approaches a certain value as $n$ becomes larger and larger, we call this value the empirical probability.

- **probability**: The probability of an event is a number that lies between 0 and 1, inclusive, that represents the likelihood of the event occurring. The larger the probability, the more likely the event is to occur.

- **simple (or elementary) event**: an event that consists of exactly one outcome

- **probability distribution**: a table that lists the probability of each simple event of an experiment

- **probability function**: the function which assigns a probability to each simple event of an experiment

- **uniform sample space**: a sample space in which all of the outcomes are equally likely

Relevant Equations:

- For a uniform sample space $S = \{s_1, s_2, ..., s_n\}$, $P(s_1) = P(s_2) = \cdots = P(s_n) = \frac{1}{n}$.

- Let $S$ be a uniform sample space and let $E$ be any event. Then,
  $$P(E) = \frac{\text{Number of outcomes in } E}{\text{Number of outcomes in } S} = \frac{n(E)}{n(S)}$$

Properties of Probability Functions:

1. $0 \leq P(s_i) \leq 1$
2. $P(s_1) + P(s_2) + \cdots + P(s_n) = 1$
3. $P(\{s_i \cup \{s_j\}) = P(s_i) + P(s_j)$ for $i \neq j$

Finding the Probability of an Event $E$:

1. Determine a sample space $S$ associated with the experiment.
2. Assign probabilities to the simple events of $S$.
3. If $E = \{s_1, ..., s_m\}$ where $\{s_1\}, ..., \{s_m\}$ are simple events then
   $$P(E) = P(s_1) + P(s_2) + \cdots + P(s_m)$$
4. If $E$ is the empty set, then $P(E) = 0$.

Rules of Probability:

1. $P(E) \geq 0$ for any event $E$.
2. $P(S) = 1$.
3. If $E$ and $F$ are mutually exclusive (that is, $E \cap F = \emptyset$), then $P(E \cup F) = P(E) + P(F)$.
4. For any events $E$ and $F$, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.
5. $P(E^C) = 1 - P(E)$.
1. Consider the experiment of flipping a coin and noting whether it lands heads or tails then rolling a six-sided die and noting the uppermost facing number.

   (a) List the simple events associated with this experiment.
   \[ S = \{ H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6 \} \]

   (b) What is the probability that a heads is flipped?
   \[ \frac{6}{12} = \frac{1}{2} \]

   (c) What is the probability that a tails is flipped and an even number is rolled?
   \[ \frac{3}{12} = \frac{1}{4} \]

2. (Tan 7.2 #5) In a survey conducted to determine whether movie attendance is increasing (i), decreasing (d), or holding steady (s) among various sectors of the population, participants are classified as follows:
   - Group 1: Those aged 10-19
   - Group 2: Those aged 20-29
   - Group 3: Those aged 30-39
   - Group 4: Those aged 40-49
   - Group 5: Those aged 50 and older

   The response and age group of each participant are recorded. List the simple events associated with this experiment.
   \[ \{ i1, i2, i3, i4, i5, d1, d2, d3, d4, d5, s1, s2, s3, s4, s5 \} \]

3. A six-sided die is weighted so that the probability of rolling each number is given by the table below.

<table>
<thead>
<tr>
<th>Roll</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>( \frac{1}{10} )</td>
<td>( \frac{1}{10} )</td>
<td>( \frac{2}{10} )</td>
<td>( \frac{3}{10} )</td>
<td>( \frac{2}{10} )</td>
<td>( \frac{1}{10} )</td>
</tr>
</tbody>
</table>

   (a) What is the probability of rolling a 2 or a 5?
   \[ P(2 \cup 5) = P(2) + P(5) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} \]

   (b) What is the probability of rolling an odd number?
   \[ P(\text{odd}) = P(1) + P(3) + P(5) = \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2} \]
4. (Tan 7.2 #16) The accompanying data were obtained from a survey of 1500 Americans who were asked: How safe are American-made consumer products?

<table>
<thead>
<tr>
<th>Rating</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Respondents</td>
<td>285</td>
<td>915</td>
<td>225</td>
<td>30</td>
<td>45</td>
</tr>
</tbody>
</table>

A: Very safe  
B: Somewhat safe  
C: Not too safe  
D: Not safe at all  
E: Don’t know

(a) Determine the empirical probability distribution associated with these data.

<table>
<thead>
<tr>
<th>Ratings</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>( \frac{285}{1500} )</td>
<td>( \frac{915}{1500} )</td>
<td>( \frac{225}{1500} )</td>
<td>( \frac{30}{1500} )</td>
<td>( \frac{45}{1500} )</td>
</tr>
</tbody>
</table>

(b) If a person who participated in the survey is selected at random, what is the probability that they responded with an A or a B?

\[
\frac{285}{1500} + \frac{915}{1500} = \frac{1200}{1500} = \frac{4}{5}
\]

5. Determine whether the sample spaces associated with the given experiments are uniform.

(a) A card is drawn at random from a standard 52-card deck and the suit is observed. **Uniform**

(b) A ball is selected at random from an urn containing 5 red balls and 7 green balls, and the color is observed. **Not uniform**

(c) Two fair six-sided dice are rolled and the number appearing uppermost on each die is observed. **Uniform**

(d) Two fair six-sided dice are rolled and the sum of the numbers appearing uppermost is observed. **Not uniform**
6. Let $E$ and $F$ be two events of an experiment with sample space $S$. Suppose $P(E) = 0.4$, $P(F) = 0.7$ and $P(E \cup F) = 0.8$. Compute:
   
   a) $P(E \cap F) = \frac{3}{4}$

   $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
   
   $0.8 = 0.4 + 0.7 - P(E \cap F)$
   
   $P(E \cap F) = 1.1 - P(E \cap F)$

   b) $P(E^C) = 1 - P(E) = 1 - 0.4 = 0.6$

   c) $P(E^C \cap F) = 0.4$

7. Let $E$ and $F$ be mutually exclusive events and suppose $P(E) = 0.3$ and $P(F) = 0.25$. Compute:
   
   a) $P(E \cup F) = P(E) + P(F) = 0.3 + 0.25 = 0.55$

   b) $P(F^C) = 1 - P(F) = 1 - 0.25 = 0.75$

   c) $P(E^C \cap F^C) = P((E \cup F)^C) = 1 - P(E \cup F)$

   $= 1 - 0.55$

   $= 0.45$

8. A card is drawn at random from a standard 52-card deck. Find the probability of the following events:
   
   a) $E$ is the event that a jack or a heart is drawn.

   $P(E) = P(J) + P(H) - P(J \cap H)$

   $= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

   b) $F$ is the event that a queen or a king is drawn.

   $P(F) = P(Q) + P(K) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$

   c) $G$ is the event that a 2 or 3 is not drawn.

   $G^c$ = event that a 2 or 3 is drawn

   $P(G) = 1 - P(G^c) = 1 - \frac{8}{52} = \frac{44}{52} = \frac{11}{13}$
9. A pair of fair six-sided dice are rolled and the numbers facing uppermost on each die are observed. Find the probability of the following events:

a) $E$ is the event that the sum of the dice is not 5.

\[ P(E^c) = 1 - P(E) = 1 - \frac{4}{36} = \frac{32}{36} = \frac{8}{9} \]

b) $F$ is the event that at least one of the dice is a 4 or at least one of the dice is a five.

\[ P(F) = P(4) + P(5) - P(4 \cap 5) = \frac{11}{36} + \frac{11}{36} - \frac{2}{36} = \frac{20}{36} = \frac{5}{9} \]

10. A test has 8 true/false questions. If a student randomly guesses on every question, what is the probability that they will get at least 4 problems correct?

- Exactly 4: \( \binom{8}{4} = \frac{70}{256} \)
- Exactly 5: \( \binom{8}{5} = \frac{56}{256} \)
- Exactly 6: \( \binom{8}{6} = \frac{28}{256} \)
- Exactly 7: \( \binom{8}{7} = \frac{8}{256} \)
- Exactly 8: \( \binom{8}{8} = \frac{1}{256} \)

\[ P(\text{at least 4}) = P(4) + P(5) + P(6) + P(7) + P(8) = \frac{70 + 56 + 28 + 8 + 1}{256} = \frac{165}{256} \]

11. A test has 8 multiple choice questions with 4 choices each. If a student randomly guesses on every question, what is the probability that they will get exactly 4 problems correct?

- 1. Which 4 correct? \( \binom{8}{4} = 70 \)
- 2. \( \frac{1 \cdot 1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 3 \cdot 3} = 3^4 = 81 \)

\[ P(4 \text{ correct}) = \binom{8}{4} \cdot \binom{4}{2} = \frac{81}{70 \times 8} = \frac{5670}{5670} = 1 \]

12. A test has 4 true/false questions and 4 multiple choice questions. If a student randomly guesses on every question, what is the probability that they will get exactly 2 true/false problems correct and exactly 2 multiple choice questions correct?

- 1. Which 2 correct? \( \binom{4}{2} \cdot \binom{4}{2} = 36 \)
- 2. \( \frac{1 \cdot 1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 3 \cdot 3} = \frac{36}{9} \)

\[ P(2 \text{ true/false, 2 multiple choice}) = \binom{4}{2} \cdot \binom{4}{2} = \frac{36}{9} \cdot \frac{324}{4096} = \frac{10625}{4096} \]

\[ P(2 \text{ true/false, 2 multiple choice}) = \frac{10625}{4096} \]
13. A five card hand is dealt from a standard 52-card deck.

\[ \binom{52}{5} = 2,598,960 \]

a) What is the probability that the hand contains a three-of-a-kind (three cards of the same rank)?

\[
\begin{align*}
\text{3 Aces} & : \frac{\binom{4}{1} \cdot \binom{48}{2}}{\binom{52}{5}} = \frac{4 \cdot 1,729}{2,598,960} = 0.02424 \\
\text{No aces} & : \frac{\binom{12}{3} \cdot \binom{48}{2}}{\binom{52}{5}} = \frac{220 \cdot 1,729}{2,598,960} = 0.02424 \\
\text{Total} & : \frac{3 \cdot \binom{4}{1} \cdot \binom{48}{2} + \binom{12}{3} \cdot \binom{48}{2}}{\binom{52}{5}} = 0.02424 \times 3 + 0.02424 = 0.07272
\end{align*}
\]

b) What is the probability that the hand is a full house (three cards of one rank and two of another)?

1. Ways we can choose two cards: \[ \frac{13 \cdot 12}{2} = 78 = \binom{13}{2} \]
2. Ways to get 3 of one and 2 of another: \[ \binom{12}{3} \cdot \binom{4}{2} = 274 \]

\[
\begin{align*}
\text{Total} & : \frac{13 \cdot 12}{2} \cdot \frac{\binom{12}{3} \cdot \binom{4}{2}}{\binom{52}{5}} = \frac{78 \cdot 274}{2,598,960} = 0.00144
\end{align*}
\]

c) What is the probability that exactly three of the cards are red?

\[
\binom{26}{3} - \binom{26}{2} = 6,552 - 351 = 6,201
\]

14. A bag of 23 Skittles contains 5 red Skittles, 4 yellow Skittles, 3 green Skittles, 5 orange Skittles, and 6 purple Skittles. A sample of 7 Skittles is chosen from the bag.

a) What is the probability that the sample contains exactly 2 purple Skittles?

\[
\binom{6}{2} \cdot \binom{17}{5} = 1,287,000
\]

\[
\frac{1,287,000}{1,610,513} \approx 0.80
\]

b) What is the probability that the sample contains at least 1 purple Skittle?

\[
P(E) = 1 - P(E^c) = 1 - \frac{\binom{17}{7}}{1,610,513} = \frac{220,755}{1,610,513} \approx 0.14
\]

b) What is the probability that the sample contains exactly 2 red Skittles or exactly 2 yellow Skittles?

\[
P(2R \cup 2Y) = P(2R) + P(2Y) - P(2R \cap 2Y) = \frac{\binom{5}{2} \cdot \binom{18}{5} + \binom{4}{2} \cdot \binom{19}{5}}{1,610,513} - \frac{\binom{5}{2} \cdot \binom{4}{2} \cdot \binom{19}{3}}{1,610,513}
\]

\[
6 \cdot \frac{85,680 + 69,750 - 21,840}{1,610,513} = \frac{133,608}{1,610,513} \approx 0.08