Math 141 Week-in-Review 9
Sections 8.1-8.4 and Review

Key Terms and Relevant Equations:

- **relative frequency:** Suppose we repeat an experiment $n$ times and an event $E$ occurs $m$ of those times. Then \( \frac{m}{n} \) is called the relative frequency of the event $E$.
- **empirical probability:** If the relative frequency approaches a certain value as $n$ becomes larger and larger, we call this value the empirical probability.
- **probability:** The probability of an event is a number that lies between 0 and 1, inclusive, that represents the likelihood of the event occurring. The larger the probability, the more likely the event is to occur.
- **simple (or elementary) event:** an event that consists of exactly one outcome
- **probability distribution:** a table that lists the probability of each simple event of an experiment
- **probability function:** the function which assigns a probability to each simple event of an experiment
- **uniform sample space:** a sample space in which all of the outcomes are equally likely
- **finite discrete:** a rule that assigns a number to each outcome of an experiment. We usually denote the random variable by $X$.
- **infinite discrete:** a rule that assigns a number to each outcome of an experiment. We usually denote the random variable by $X$.
- **continuous:** a graphical representation of a probability distribution. The x-axis of a histogram corresponds to the values of the random variable and the y-axis corresponds to probabilities. Normally, we draw a bar of width one centered above each value of the random variable and the height of the bar is the probability that the random variable takes on that particular value.
- **mean:** The average or mean of the $n$ numbers $x_1, x_2, \ldots, x_n$ denoted by $\overline{x}$ or $\mu$ is given by $\mu = \frac{x_1 + x_2 + \cdots + x_n}{n}$.
- **expected value:** Let $X$ denote the random variable that has values $x_1, x_2, \ldots, x_n$, and let the associated probabilities be $p_1, p_2, \ldots, p_n$. The expected value of the random variable $X$, denoted by $E(X)$, is $E(X) = x_1p_1 + x_2p_2 + \cdots + x_n p_n$. 

\[ P(E) = \frac{\text{Number of outcomes in } E}{\text{Number of outcomes in } S} = \frac{n(E)}{n(S)} \]

The conditional probability that the event $F$ will occur given that the event $E$ occurs is $P(F|E) = \frac{P(E \cap F)}{P(E)}$.

Product Rule: $P(E \cap F) = P(E)P(F|E)$.

**Independent:** Two events $E$ and $F$ are said to be independent if $P(E|F) = P(E)$ and $P(F|E) = P(F)$. In other words, two events are independent if the outcome of one does not affect the outcome of the other.

Two events $E$ and $F$ are independent if $P(E \cap F) = P(E)P(F)$

$n$ events $E_1, E_2, \ldots, E_n$ are independent if $P(E \cap E_2 \cap \cdots \cap E_n) = P(E_1)P(E_2) \cdots P(E_n)$

**Bayes’ Theorem:**

\[ P(E|F) = \frac{P(E)P(F|E)}{P(E)P(F|E) + P(E^c)P(F|E^c)} \]

**random variable:** a rule that assigns a number to each outcome of an experiment. We usually denote a random variable by $X$.

**finite discrete random variable:** A random variable is finite discrete if it assumes only a finite number of values.

**infinite discrete random variable:** A random variable is infinite discrete if it takes on an infinite number of values that can be listed in a sequence, so that there is a first one, a second one, a third one, and so on.

**continuous random variable:** A random variable is said to be continuous if it can take any of the infinite number of values in some interval of real numbers.

**histogram:** a graphical representation of a probability distribution. The x-axis of a histogram corresponds to the values of the random variable and the y-axis corresponds to probabilities. Normally, we draw a bar of width one centered above each value of the random variable and the height of the bar is the probability that the random variable takes on that particular value.
4.) The trials are independent of each other.
2.) There are two possible outcomes of each trial: success and failure.
1.) The number of trials in the experiment is fixed.

- **fair game**: In a game situation, let X be the net winnings (profit) of the player. A game is considered “fair” when the expected net winnings are 0, i.e., when \( E(X) = 0 \).

- **odds in favor**: If \( P(E) \) is the probability of an event \( E \) occurring, then the odds in favor of \( E \) occurring are \( \frac{P(E)}{1-P(E)} \) or \( \frac{P(E)}{P(E^C)} \) (if \( P(E) \neq 1 \)).

- **odds against**: If \( P(E) \) is the probability of an event \( E \) occurring, then the odds against \( E \) occurring are \( \frac{1-P(E)}{P(E)} \) or \( \frac{P(E^C)}{P(E)} \) (if \( P(E) \neq 0 \)).

- **Obtaining Probability from Odds**: Suppose that the odds in favor of an event \( E \) occurring are \( a \) to \( b \), then \( P(E) = \frac{a}{a+b} \).

- **median**: The median of a group of numbers arranged in increasing or decreasing order is (a) the middle number if there is an odd number of entries or (b) the mean of the two middle numbers if there is an even number of entries.

- **mode**: The mode of a group of numbers is the number in the group that occurs most frequently.

- **variance**: Let \( X \) denote the random variable that takes on the values \( x_1, x_2, ..., x_n \), and let the associated probabilities be \( p_1, p_2, ..., p_n \). Then if \( \mu = E(X) \), the variance of the random variable, denoted by \( Var(X) \), is \( Var(x) = p_1(x_1 - \mu)^2 + p_2(x_2 - \mu)^2 + \cdots + p_n(x_n - \mu)^2 \).

- **standard deviation**: The standard deviation of a random variable \( X \), denoted by \( \sigma \), is defined by \( \sigma = \sqrt{Var(X)} \).

- **Bernoulli trial**: An experiment with two outcomes is called a Bernoulli (or binomial) trial.

- **binomial experiment**: An experiment that consists of repeated Bernoulli trials.

Let \( X \) be the random variable that represents the number of successes of a binomial experiment with \( n \) trials and probability of success \( p \) for each trial. Then the probability of \( k \) successes is \( P(X = k) = \binom{n}{k} p^k q^{n-k} \) where \( q \) is the probability of failure.

- **Calculating binomial probabilities in your calculator**: \( \text{binompdf}(n, p, k) \) calculates \( P(X = k) \) and \( \text{binomcdf}(n, p, k) \) calculates \( P(X \leq k) \).

If \( X \) is a binomial random variable associated with a binomial experiment consisting of \( n \) trials with probability of success \( p \) and probability of failure \( q \), then the mean (expected value), variance, and standard deviation are: \( E(X) = np \), \( Var(X) = npq \) and \( \sigma = \sqrt{npq} \).

**Properties of Probability Functions:**

1. \( 0 \leq P(s_i) \leq 1 \)
2. \( P(s_1) + P(s_2) + \cdots + P(s_n) = 1 \)
3. \( P(\{s_i \cup \{s_j\}) = P(s_i) + P(s_j) \) for \( i \neq j \)

**Rules of Probability:**

1. \( P(E) \geq 0 \) for any event \( E \).
2. \( P(S) = 1 \).
3. If \( E \) and \( F \) are mutually exclusive (that is, \( E \cap F = \emptyset \)), then \( P(E \cup F) = P(E) + P(F) \).
4. For any events \( E \) and \( F \), \( P(E \cup F) = P(E) + P(F) - P(E \cap F) \).
5. \( P(E^C) = 1 - P(E) \).

**Properties of Binomial Experiments:**

1.) The number of trials in the experiment is fixed.
2.) There are two possible outcomes of each trial: success and failure.
3.) The probability of success in each trial is the same.
4.) The trials are independent of each other.
1. Classify the following random variables.

(a) Cards are drawn, one at a time and without replacement, from a standard 52-card deck until a heart is drawn. \(X\) is the number of cards that are drawn.
\[
\{1, 2, 3, \ldots, 40\} \quad \text{finite discrete}
\]

(b) Cards are drawn, one at a time and with replacement, from a standard 52-card deck until a king is drawn. \(X\) is the number of cards that are drawn.
\[
\{1, 2, 3, \ldots\} \quad \text{infinite discrete}
\]

(c) A shot put is thrown, and \(X\) is the distance it travels measured in feet.
\(X \geq 0 \quad \text{continuous}\)

2. The probability distribution of the random variable \(X\) is shown in the table.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(X = x))</td>
<td>0.12</td>
<td>0.18</td>
<td>0.29</td>
<td>0.11</td>
<td>0.08</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Find:

a) \(P(X = 4) = \boxed{0.29}\)

b) \(P(X \leq 6) = 0.12 + 0.18 + 0.29 + 0.11 = \boxed{0.7}\)

c) \(P(4 \leq X < 10) = 0.29 + 0.11 + 0.08 = \boxed{0.48}\)

d) \(P(X = 3) = \boxed{0}\)

3. Three cards are drawn without replacement from a standard 52-card deck. Let \(X\) denote the number of spades drawn. Find the probability distribution for \(X\). Then draw a histogram.

\[
P(X = 0) = \frac{C(39, 3)}{C(52, 3)} = \frac{9139}{22100} = \frac{703}{1700}
\]

\[
P(X = 1) = \frac{C(13, 1) \cdot C(39, 2)}{C(52, 3)} = \frac{9633}{22100} = \frac{741}{1700}
\]

\[
P(X = 2) = \frac{C(13, 2) \cdot C(39, 1)}{C(52, 3)} = \frac{3042}{22100} = \frac{117}{850}
\]

\[
P(X = 3) = \frac{C(13, 3)}{C(52, 3)} = \frac{286}{22100} = \frac{11}{850}
\]
4. The following is a list of the amount of rainfall in Seattle for each day of September (measured in inches). Find the mean, median, mode, and standard deviation.

\[0, 0.10, 0.02, 0.01, 0.94, 0.32, 0.01, 0.02, 0, 0, 0, 0, 0, 0, 0, 0.39, 0, 0, 0, 0, 0, 0.16, 0.01, 0.39, 0.17, 0.07, 0.07, 0.01, 0.12, 1.11, 0.82, 0.91\]

\[\mu = \bar{x} = 0.1883\]
\[\text{Med} = 0.015\]
\[\text{Mode} = 0\]
\[\sigma = 0.3188\]

5. The following histogram represents the probability distribution of a random variable \(X\).

Find the expected value, variance, and standard deviation of \(X\).

\[E(X) = 1 \cdot 0.2 + 2 \cdot 0.4 + 3 \cdot 0.1 + 4 \cdot 0.2 + 5 \cdot 0.1 = 2.6\]

\[\text{Var}(x) = 0.2 \cdot (1 - 2.6)^2 + 0.4 \cdot (2 - 2.6)^2 + 0.1 \cdot (3 - 2.6)^2 + 0.2 \cdot (4 - 2.6)^2 + 0.1 \cdot (5 - 2.6)^2 = \sqrt{1.64} = 1.2806\]

\[\sigma = \sqrt{1.64} = 1.2806\]
6. Visitors at a carnival pay $a$ to play a game. The game consists of pulling 3 balls out of a bag. The bag contains 15 balls. 5 of the balls are green, 7 are red, and 3 are blue. If one of the three balls is green, the player wins $1$. If two of the balls are green, the player wins $2$. If all three of the balls are green, the player wins $3$. Otherwise, the player wins nothing. Let $X$ be a random variable whose value is the net winnings of a player.

(a) Find the probability distribution for $X$.

<table>
<thead>
<tr>
<th>$X$</th>
<th>-a</th>
<th>1-a</th>
<th>2-a</th>
<th>3-a</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X=x)$</td>
<td>$\frac{24}{91}$</td>
<td>$\frac{45}{91}$</td>
<td>$\frac{20}{91}$</td>
<td>$\frac{2}{91}$</td>
</tr>
</tbody>
</table>

- $P(X=-a) = \frac{C(10,3)}{C(15,3)} = \frac{120}{455} = \frac{24}{91}$
- $P(X=1-a) = \frac{C(5,1) \cdot C(10,2)}{C(15,3)} = \frac{45}{91}$
- $P(X=2-a) = \frac{C(5,2) \cdot C(10,1)}{C(15,3)} = \frac{20}{91}$
- $P(X=3-a) = 1 - \frac{24}{91} - \frac{45}{91} - \frac{20}{91} = \frac{2}{91}$

(b) What is the minimum amount that you would expect to pay to play this game? (Hint: it is the amount that makes it a fair game)

$$E(X) = 0$$
$$E(X) = \frac{24}{91}(-a) + \frac{45}{91}(1-a) + \frac{20}{91}(2-a) + \frac{2}{91}(3-a)$$

$$= -\frac{24}{91}a + \frac{45}{91} - \frac{45}{91}a + \frac{20}{91}a + \frac{2}{91}a$$

$$= -a + 1$$

7. In the casino game roulette, a ball is spun on a spinning wheel which contains 38 numbered spaces. If a player bets on a specific number, then they win if the ball lands on that number and lose if it does not.

(a) Mitch places a bet on the number 17 space. Assuming that the ball has an equal chance of landing on any of the spaces, what are the odds in favor of Mitch winning?

$$\text{Prob: } \frac{1}{38} \quad \text{Odds: } \frac{\frac{1}{38}}{\frac{37}{38}} = \frac{1}{37} \quad \text{or } \frac{1}{38} \cdot 37 \quad \text{or } 1:37$$

(b) If Mitch’s bet was $5, what must the payout be to make it a fair game?

$$E(X) = \frac{37}{38}(-5) + \frac{1}{38}(a-5)$$

$$= -\frac{185}{38} + \frac{a}{38} - \frac{5}{38}$$

$$= \frac{a}{38} - 5$$

$$\frac{a}{38} - 5 = 0$$

$$\frac{a}{38} = 5$$

$$a = 190$$

Actual payout $\$190$
8. Determine if the following are binomial experiments.

(a) Rolling a fair six-sided die ten times and observing the number that appears uppermost.
   \text{No, more than 2 outcomes per trial.}

(b) Rolling a fair six-sided die ten times and observing the number of times a 4 is rolled.
   \text{Yes}

(c) Rolling a fair six-sided die until a 4 is rolled.
   \text{No. Not a fixed # of trials.}

(d) Recording whether an accident occurs at a given intersection on four clear days and one rainy day.
   \text{No, not same probability}

9. Three cards are drawn one at a time and with replacement from a standard 52-card deck. Let \( X \) denote the number of spades drawn. Find the probability distribution for \( X \).

\[
\begin{align*}
  n &= 3 \\
  p &= \frac{13}{52} = \frac{1}{4}
\end{align*}
\]

\[
\begin{align*}
  P(X=0) &= \text{binompdf}(3, \frac{1}{4}, 0) = 0.4219 \\
  P(X=1) &= \text{binompdf}(3, \frac{1}{4}, 1) = 0.4219 \\
  P(X=2) &= \text{binompdf}(3, \frac{1}{4}, 2) = 0.1406 \\
  P(X=3) &= \text{binompdf}(3, \frac{1}{4}, 3) = 0.0156
\end{align*}
\]

10. A certain company employs 123 women and 97 men. Each month, one employee is selected randomly to be the “employee of the month” and is given the privilege of parking in a special reserved parking space. Each employee is eligible to win every month.

\[
\begin{align*}
  n &= 12 \\
  p &= \frac{123}{220}
\end{align*}
\]

(a) In the year 2016, what is the probability that exactly 6 women are chosen?

\[
\text{binompdf}(12, \frac{123}{220}, 6) = 0.2073
\]

(b) In the year 2016, what is the probability that at most 9 women are chosen?

\[
\text{binomcdf}(12, \frac{123}{220}, 9) = 0.9520
\]
(c) In the year 2016, what is the probability that at least 3 women are chosen?

\[
P(\text{at least 3}) = 1 - P(\text{at most 2}) = 1 - \text{binomcdf}(12, \frac{123}{220}, 2) = 0.9934
\]

(d) In the year 2016, what is the probability that between 4 and 8 woman are chosen, inclusive?

\[
\text{binomcdf}(12, \frac{123}{220}, 8) - \text{binomcdf}(12, \frac{123}{220}, 3) = 0.8205
\]

(e) What is the expected number of women chosen?

\[
E(X) = np = 12 \cdot \frac{123}{220} = \frac{369}{55} \approx 6.7091
\]

Section 7.2: Definition of Probability

11. A biased die was rolled repeatedly, and the results of the experiment are given in the following table.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>120</td>
<td>140</td>
<td>155</td>
<td>110</td>
<td>135</td>
<td>140</td>
</tr>
<tr>
<td>Total</td>
<td>800</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) Find the empirical probability distribution for this experiment.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>(\frac{120}{800})</td>
<td>(\frac{140}{800})</td>
<td>(\frac{155}{800})</td>
<td>(\frac{110}{800})</td>
<td>(\frac{135}{800})</td>
<td>(\frac{140}{800})</td>
</tr>
</tbody>
</table>

(b) Find the probability of the event \(A = \{1, 2, 5\}\).

\[
P(A) = \frac{120}{800} + \frac{140}{800} + \frac{135}{800} = \frac{395}{800} = \frac{79}{160}
\]

(c) Find the probability of the event \(B = \{3, 4, 5\}\).

\[
P(B) = \frac{155}{800} + \frac{110}{800} + \frac{135}{800} = \frac{400}{800} = \frac{1}{2}
\]

(d) Find the probability of the event \(C = \{1, 2, 3, 4, 5, 6\}\).

\[
P(C) = 1
\]
Section 7.3: Rules of Probability

12. John and Sarah are playing the board game Settlers of Catan. In each round of the game, two fair six-sided dice are rolled and each player may receive resources based on the resulting roll. John receives a resource when the sum of the dice is a 3, 5, or 9 and Sarah receives a resource when the sum of the dice is a 3, 8, or 12.

\[ \text{sum: } 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \]
\[ \# \text{rolls: } 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1 \]

a) In any given round, what is the probability that John receives a resource?

\[ P(J) = \frac{2}{36} + \frac{4}{36} + \frac{4}{36} = \frac{10}{36} = \frac{5}{18} \]

b) Which player is more likely to receive a resource?

\[ P(S) = \frac{2}{36} + \frac{5}{36} + \frac{1}{36} = \frac{8}{36} = \frac{2}{9} \]

John is more likely

c) What is the probability that both players receive a resource in the same round?

\[ P(J \cap S) = \frac{2}{36} = \frac{1}{18} \]

d) What is the probability that at least one of the two players receive a resource?

\[ P(J \cup S) = P(J) + P(S) - P(J \cap S) = \frac{5}{18} + \frac{2}{9} - \frac{1}{18} = \frac{4}{9} \]

Section 7.4: Use of Counting Techniques in Probability

13. Taylor buys a bag of sour skittles. The bag contains 9 red skittles, 7 green skittles, 4 purple skittles, 10 orange skittles, and 5 yellow skittles. She reaches in the bag and pulls out 5 skittles at random.

\[ \text{35 total} \]

(a) Red skittles are her favorite. What is the probability that at least one of her five skittles is red?

\[ P(\text{at least 1 red}) = 1 - P(\text{0 red}) \]
\[ = 1 - \frac{C(26,5)}{C(35,5)} = 1 - \frac{65780}{324632} = \frac{5883}{7378} \]

(b) What is the probability that exactly 2 skittles are green?

\[ \frac{C(7,2) \cdot C(28,3)}{C(35,5)} = \frac{168796}{324632} \]
Section 7.5: Conditional Probability and Independent Events

14. A pair of fair six-sided dice are rolled. Let $E$ denote the event that the number falling uppermost on the first die is a 4, and let $F$ denote the event that the sum of the numbers falling uppermost is 6.

(a) Compute $P(E)$.
\[ \frac{6}{36} = \frac{1}{6} \]

(b) Compute $P(E \cap F)$.
\[ \frac{1}{36} \]

(c) Compute $P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{\frac{4}{36}}{\frac{1}{6}} = \frac{1}{6}$

(d) Compute $P(F)$
\[ \frac{5}{36} \]

(e) Are $E$ and $F$ independent events?

No because $P(F|E) \neq P(F)$.

Check: $P(E \cap F) = P(E) \cdot P(F)$?
\[ \frac{1}{36} \neq \frac{1}{6} \cdot \frac{5}{36} \]
Section 7.6: Bayes Theorem

15. A study of the faculty at U.S. medical schools in 2006 revealed that 32% of the faculty were women and 68% were men. Of the female faculty, 31% were full professors, 47% were assistant professors, and 22% were instructors. Of the male faculty, 51% were full professors, 37% were assistant professors, and 12% were instructors. If a faculty member at a U.S. medical school in 2006 was selected at random,

(a) What is the probability that the faculty member was a woman?

\[ \boxed{0.32} \]

(b) What is the probability that the faculty member was a full professor?

\[ 0.32 \cdot 0.31 + 0.68 \cdot 0.51 = \boxed{0.446} \]

(c) What is the probability that the faculty member was an assistant professor if it is known that they were a man?

\[ P(A | M) = \boxed{0.37} \]

(d) What is the probability that the faculty member was a woman if it is known that they are an instructor?

\[ P(W | I) = \frac{P(W \cap I)}{P(I)} = \frac{0.32 \cdot 0.22}{0.32 \cdot 0.22 + 0.68 \cdot 0.12} = \frac{0.0704}{0.152} = \boxed{\frac{44}{95}} \]