NUMERICAL ANALYSIS QUALIFIER
January 12, 2009

Problem 1. Let $A$ and $B$ be matrices in $\mathbb{R}^{n \times n}$, $A$ be non-singular, and satisfy the inequality $\|A^{-1}\|_2\|B\|_2 \leq q$ with a constant $q < 1$. Here $\| \cdot \|_2$ is the matrix norm subordinate to the Euclidean norm in $\mathbb{R}^n$.

(a) Show that $C = A + B$ is non-singular.
(b) Show that the iteration process $Ax^{j+1} = b - Bx^j$, $j = 0,1,\ldots$ converges for any $x^0$ to the solution of the system $Cx = b$. Give an estimate for the Euclidean norm of error $x^j - x$ in terms of $q$.
(c) Let $A = 2I$, where $I$ is the identity matrix in $\mathbb{R}^{n \times n}$, and let $B$ be the matrix with entries of $-1$ on the two main co-diagonals and zeros elsewhere. Compute $\|A^{-1}\|_2\|B\|_2$ in terms of $n$.

Problem 2. Let $P^n$ denote the space of polynomials of degree less than or equal to $n$. Let $\|f\|_2^2 = \int_0^1 |f(x)|^2 \, dx$ and define $\Pi_n f$ to be the best approximation to $f$ in $P^n$ in the norm $\| \cdot \|_\omega$, i.e.

$$\|\Pi_n f - f\|_\omega = \min_{\phi \in P_n} \|f - \phi\|_\omega.$$ 

Finally, let $T_k(x)$ denote the Chebyshev polynomial of order $k$. You may use the fact that $T_k$ is $\omega$-orthogonal to $P^{k-1}$.

(a) Give a formula for $\Pi_n f$ in terms of the Chebyshev polynomials $\{T_k(x)\}_{k=0}^n$.
(b) Let $f \in P^{n+1}$. Show that

$$\|f - \Pi_n f\|_\infty = \inf_{q \in P_n} \|f - q\|_\infty.$$ 

Problem 3. Consider the initial value problem for the ordinary differential equation

$$y' = f(y,t), \quad y(t_0) = y_0$$

and a one-step numerical method of the form

$$u_{n+1} = u_n + h \, M \left( u_n, t_n \right),$$

where $u_0 = y_0$, $h \in (0,1)$ is the step size, $u_n \approx y(t_n)$, and $t_{n+1} = t_n + h$.

(a) Rewrite the extrapolated method defined by

$$u_{n+1} = u_n + h \, M \left( u_n, t_n \right),$$

as a one-step method and find all values of $\alpha$ and $\beta$ for which the above scheme is consistent?
(b) Find the values of $\alpha$ and $\beta$ which make the above method second order.
(c) Prove or disprove: The method of Part (b) is absolutely stable (A-stable).

Problem 4. Let $\Omega = (0,1)$ and $u$ be the solution of the second order hyperbolic problem:

$$u_{tt} - u_{xx} = 0 \quad \text{for} \quad (x,t) \in \Omega \times (0,T),$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{for} \quad x \in \Omega,$n$$

$$u(0,t) = 0, \quad u(1,t) = 0 \quad \text{for} \quad t \in (0,T).$$
(a) Describe the finite difference scheme for this initial value problem which results when the spatial derivative is approximated by the three point stencil at time level \( n \) and the second order partial derivative with respect to time is discretized by the central difference:

\[
\begin{align*}
  u_{xx}(x_i, t_n) &\approx \frac{U^n_{i-1} - 2U^n_i + U^n_{i+1}}{h^2} \\
  u_{tt}(x_i, t_n) &\approx \frac{U^{n+1}_i - 2U^n_i + U^{n-1}_i}{k^2}.
\end{align*}
\]

Here \( k \) and \( h = \frac{1}{N+1} \) are the step sizes in time and space (respectively) and \( U^n_i \) approximates the solution \( u(x_i, t_n) \) where \( t_n = nk \) and \( x_i = ih, \ n = 0, 1, \ldots, \ i = 0, 1, \ldots, N \). Make sure that you are explicit about boundary conditions and the initial two steps \( (U^0 \text{ and } U^1) \) for this discretization.

(b) Estimate the local truncation error of the scheme.

(c) Derive a Courant condition for stability in time for the scheme of Part (a).

**Problem 5.** Let \( \Omega \) be a domain in \( \mathbb{R}^2 \) and \((\cdot, \cdot)\) denote the inner product in \( L^2(\Omega) \). Let \( A(\cdot, \cdot) \) be a (nonsymmetric) bounded bilinear form on \( H^1_0(\Omega) \times H^1_0(\Omega) \) satisfying the Gårding inequality:

\[
\|w\|_{H^1(\Omega)}^2 - c_0\|w\|_{L^2(\Omega)}^2 \leq c_1 A(w, w) \quad \text{for all } w \in H^1_0(\Omega)
\]

(\( c_0, c_1 \) are positive constants). For \( g \in L^2(\Omega) \), assume that there is a unique \( v \in H^1_0(\Omega) \) satisfying

\[
A(\phi, v) = (\phi, g) \quad \text{for all } \phi \in H^1_0(\Omega).
\]

Assume further that the above problem is \( H^2 \)-regular, i.e., \( v \in H^2(\Omega) \) and satisfies

\[
\|v\|_{H^2(\Omega)} \leq c\|g\|_{L^2(\Omega)}.
\]

(a) Let \( S_h \) be a sequence of finite element spaces contained in \( H^1_0(\Omega) \) with approximation parameter \( h \in (0, 1) \). Given \( w \in H^1_0(\Omega) \), suppose that \( \theta \in S_h \) solves

\[
A(\theta, \chi) = A(w, \chi) \quad \text{for all } \chi \in S_h.
\]

Show that

\[
\|w - \theta\|_{L^2(\Omega)} \leq C h \|w - \theta\|_{H^1(\Omega)}.
\]

(b) Use (5.1) and (5.2) to show that there is an \( h_0 > 0 \) such that if \( h \leq h_0 \),

\[
\|w - \theta\|_1^2 \leq CA(w - \theta, w - \theta).
\]

Taking \( w = 0 \) above shows that for \( h \leq h_0 \), \( \eta = 0 \) is the only solution \( \eta \in S_h \) satisfying

\[
A(\eta, \chi) = 0 \quad \text{for all } \chi \in S_h.
\]

(c) Use (b) above to show that for \( h \leq h_0 \) there exists a unique solution \( u_h \in S_h \) to

\[
A(u_h, \chi) = (f, \chi) \quad \text{for all } \chi \in S_h,
\]

i.e., the finite element approximation exists and is unique.