NUMERICAL ANALYSIS QUALIFIER

May, 2007

Do all of the problems below. Show the whole work.

Problem 1. Consider the iterative method
\[ x_{k+1} = Mx_k + b, \]
where \( x_k \) and \( b \) are vectors in \( \mathbb{R}^n \), \( M \) is an \( n \times n \) real matrix, and \( x_0 \) is a given initial iterate. Assume that \( \|M\| < 1 \) where \( \|M\| \) is the induced matrix norm from a vector norm \( \|x\| \).

(a) Show that the matrix \( I - M \) is nonsingular and therefore the linear system \( x = Mx + b \) has unique solution \( x \in \mathbb{R}^n \).
(b) Show also that
\[ \|x_k - x\| \leq \|M\|^k \|x_0\| + \|M\|^k \|b\| \]
so that the above iteration converge to the solution of \( x = Mx + b \) for any initial iterate \( x_0 \).
(c) Prove that
\[ \|x_k - x\| \leq \|(I - M)^{-1}\| \|x_{k+1} - x_k\|. \]

Problem 2. Consider the initial value problem
\[ \frac{dy}{dt} = f(y,t), \quad t > t_0, \quad y(t_0) = y_0 \]
and its approximation by the Runge-Kutta method: For \( n = 0, 1, \ldots \),
\[ u_{n+1} = u_n + \frac{h}{4} (k_1 + 3k_2), \quad u_0 = y_0, \quad t_{n+1} = t_n + h, \]
where \( k_1 = f(u_n, t_n) \) and \( k_2 = f(u_n + \frac{2}{3}hk_1, t_n + \frac{2}{3}h) \). Here \( u_n \) approximates \( y(t_n) \) and \( h \) is the step-size.

(a) Use Taylor’s Theorem to show that the method is at least of order two.
(b) Define what it means for \( \eta = \lambda h \) to be in the region of absolute stability of the above scheme. Here \( \lambda \) is an arbitrary complex number.
(c) Derive an inequality which can be used to determine when \( \eta = \lambda h \) is in the region of absolute stability for this scheme.
(d) Determine which real values of \( \eta \) are in the region of absolute stability.

Problem 3. Let \( w \) be a positive integrable function on the interval \( [a, b] \) such that \( w(x) \geq w_0 > 0 \) for all \( x \in [a, b] \). Let \( \mathcal{P}_n \) be the set of polynomials on \( [a, b] \) of degree at most \( n \). Let \( A_0, \ldots, A_n, \ x_0, \ldots, x_n \) be real numbers such that \( x_i \in [a, b], \ i = 0, \ldots, n \) and
\[ \int_a^b f(t)w(t)dt = \sum_{i=0}^n A_if(x_i), \quad \forall f \in \mathcal{P}_{2n+1}. \]
(a) Show that the polynomial \( \pi_n(x) := \Pi_{i=0}^{n}(x-x_i) \) is orthogonal to \( P_n \) with respect to the inner product \( (p,q)_w := \int_a^b p(t)q(t)w(t)dt \).

(b) Prove that the coefficients \( A_i \) are all positive.

(c) Assume \( f \in C^{(2n+2)}[a,b] \). Derive a representation for

\[
E = \int_a^b f(t)w(t)dt - \sum_{i=0}^{n} A_i f(x_i)
\]

in terms of \( f^{(2n+2)} \).

**Problem 4.** Consider the boundary value problem

\[
\begin{align*}
\frac{d^4u}{dx^4}(x) &= f(x), & 0 < x < 1, \\
u(0) &= 0, & u''(0) = 0, \\
u'(1) + u''(1) &= \beta, & -u'''(1) = \gamma,
\end{align*}
\]

where \( f(x) \) is a given function on \((0,1)\) and \( \beta \) and \( \gamma \) are given constants.

(a) Give the weak formulation of this problem in an appropriate space \( V \) and characterize \( V \).

(b) Show that the corresponding linear form is coercive and continuous in \( V \) and the linear form is continuous in \( V \).

(c) Set up a finite dimensional space \( V_h \subset V \) of piece-wise polynomial functions over a uniform partition of \((0,1)\). Define the "nodal" basis in terms of the degrees of freedom.

(d) Introduce the Galerkin method for the problem (4.1) for \( V_h \); state the error estimate in the \( V \)-norm assuming smooth solution \( u(x) \).

**Problem 5.** Consider the boundary value problem: find \( u(x) \) such that

\[
\begin{align*}
-\Delta u + \alpha \frac{\partial u}{\partial x_1} + \beta x_1 \frac{\partial u}{\partial x_2} &= f(x), & x := (x_1,x_2) \in \Omega \\
u(x) &= 0, & x \in \partial \Omega.
\end{align*}
\]

Here \( \Omega \) is a bounded convex polygonal domain in \( \mathbb{R}^2 \), \( \alpha \) and \( \beta \) are given constants, and \( f(x) \) is a given smooth function. These guarantee full regularity of the solution for any \( \alpha \) and \( \beta \), i.e. \( u \in H^2(\Omega) \) and \( \|u\|_{H^2} \leq C\|f\|_{L^2} \).

(a) Derive a weak form of this problem in an appropriate space \( V \) (describe the space !).

(b) Show that the corresponding form is coercive in the norm of the space \( V \).

(c) Set up a finite element approximation on triangulation of \( \Omega \); describe the space \( V_h \subset V \) of piece-wise linear finite elements.

(d) Write down the a priori error estimate in \( V \)-norm; using Aubin-Nitsche (duality) argument derive an error estimate in the \( L^2 \)-norm.