Problem 1. Suppose that $L_1$ and $L_2$ are symmetric and positive definite $n \times n$ matrices. We consider the alternating direction implicit (ADI) iteration
\[
(I + kL_1)(I + kL_2)x_{i+1} = (I - kL_1)(I - kL_2)x_i + 2kb
\]
for solving the matrix equation $(L_1 + L_2)x = b$. Here $k > 0$ is an iteration parameter.

(a) Show that this method is consistent and find the reduction matrix associated with this iteration.

(b) Consider the norm $\|x\| = \|(I + kL_2)x\|$ and let $e_k = x - x_k$. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. Show that $\|e_{i+1}\| \leq \gamma \|e_i\|$ for some $\gamma < 1$ depending on $k$ and the spectrum of $L_1$ and $L_2$. Do not assume that $L_1$ and $L_2$ commute. (Hint: write $\|e_{i+1}\| = \|T(I + kL_2)e_i\|$ and bound $\|T\|$.)

(c) Suppose that the spectrum of $L_1$ and $L_2$ is contained in $[\mu_1, \mu_2]$ with $\mu_1 > 0$. Choose a value of $k$ which results in the optimal convergence rate. What is the resulting rate?

Problem 2. Consider the initial value problem for $y(t)$:
\[
\frac{dy}{dt} = f(y, t), \quad t > t_0, \quad y(t_0) = y_0
\]
and its approximation by the Runge-Kutta method: for $n = 0, 1, \ldots$,
\[
u_{n+1} = u_n + \frac{h}{4}(k_1 + 3k_2), \quad u_0 = y_0, \quad t_{n+1} = t_n + h,
\]
where $k_1 = f(u_n, t_n)$ and $k_2 = f(u_n + \frac{2}{3}hk_1, t_n + \frac{2}{3}h)$. Here $u_n$ approximates $y(t_n)$ and $h$ is the step-size.

(a) Use Taylor’s Theorem to show that the method is at least of order two.

(b) Define what it means for $\eta = \lambda h$ to be in the region of absolute stability of the above scheme (here $\lambda$ is an arbitrary complex number). Derive an inequality which characterizes the region of absolute stability for (2.1).

(c) Determine which real values of $\eta$ are in the region of absolute stability of (2.1).

Problem 3. Consider the boundary value problem
\[
u^{(4)}(x) + q(x)u = f(x), \quad 0 < x < 1,
u(0) = 0, \quad u(1) = 0,
u''(0) = -\gamma, \quad u'(1) + u''(1) = \beta,
\]
where $f(x)$ is a given function on $(0, 1)$, $\beta$ and $\gamma$ are given constants and $q(x) \geq 0$.

(a) Give a weak formulation of this problem in an appropriate space $V$, characterize $V$, and prove that the corresponding linear form is coercive on $V$.

(b) Set up a finite dimensional space $V_h \subset V$ of piece-wise cubic functions over a uniform partition of $(0, 1)$. Introduce the Galerkin finite element method for the problem (3.1) for $V_h$. State an error estimate in $V$-norm assuming that $u(x) \in H^4(0, 1)$ (do NOT prove this).
(c) Assuming “full regularity” and using duality argument prove the following estimate for the error of the Galerkin solution \( u_h \):

\[
\| u - u_h \|_{L^2} \leq C h^4 \| u^{(4)} \|_{L^2}.
\]

Further prove the estimate \( \| u' - u'_h \|_{L^2} \leq C h^3 \| u^{(4)} \|_{L^2} \).

**Problem 4.** Let \( \tau \) and \( \hat{\tau} = \{ x : 0 < \hat{x}_1 < 1, 0 < \hat{x}_2 < 1 - \hat{x}_2 \} \) be a triangle and reference triangle in \( \mathbb{R}^2 \), correspondingly, and let \( T_\tau \) be an affine transformation mapping \( \hat{\tau} \) to \( \tau \). Denote by \( |\tau| \) the area of \( \tau \). Let \( m_1, m_2, \) and \( m_3 \) be the mid-points of the three edges of \( \tau \).

(a) Prove that the following quadrature is exact for all polynomials in \( P_2 \):

\[
\int_\tau v(x)dx \approx \frac{1}{3}|\tau|(v(m_1) + v(m_2) + v(m_3)).
\]

(b) In part (c) you will need an inequality of the following type: there is a constant \( c(\hat{\tau}) \) so that for all \( v \in H^s(\tau) \) and \( s \geq 0 \) an integer:

\[
|\hat{v}|_{H^s(\hat{\tau})} \leq c(\hat{\tau}) h_\tau^s |\tau|^{-\frac{1}{2}} |v|_{H^s(\tau)} \text{ with } \hat{v}(\hat{x}) = v(T_\tau(\hat{x})).
\]

Prove this inequality for \( s = 1 \).

(c) Let \( h_\tau \) be the diameter of \( \tau \). Prove that there exists \( c > 0 \) (depending only on the reference triangle \( \hat{\tau} \)) so that

\[
\forall v \in H^3(\tau), \quad \left| \int_\tau v(x)dx - \frac{1}{3}|\tau|(v(m_1) + v(m_2) + v(m_3)) \right| \leq ch_\tau^3 |\tau|^{-\frac{1}{2}} |v|_{H^3(\tau)}.
\]

Hint. Use Bramble-Hilbert Lemma.

**Problem 5.** Consider the boundary value problem: find \( u(x) \) such that

\[
-\Delta u + b \frac{\partial u}{\partial x_1} + u = f \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{on } \partial\Omega.
\]

Here \( \Omega = (0, 1) \times (0, 1) \), \( b \) is a given constant, and \( f \) is a given smooth function.

(a) Derive a up-wind finite difference approximation of this problem on a uniform square mesh.

(b) Prove an \textit{a priori} estimate for the solution in the maximum norm.

(c) Show that the local truncation error is \( O(h) \) and using the \textit{a priori} estimate of part (b) show \( O(h) \)-convergence rate in maximum norm.