Notes for **Thursday** Oct 25 class C. Hillar
Math 152
Tu, Th: 8:00-9:15 Room: Held 107

Section 10.1 Stewart Calculus

Recall a sequence is a list of numbers

$$a_1, a_2, a_3, \ldots.$$

Usually, we write this list as the symbol $\{a_n\}$. e.g. the sequence $a_n = 1/2^n$

$$½, ¼, 1/8, 1/16, \ldots$$

is written $\{1/2^n\}$. Plotting this sequence, we see that $a_n = 1/2^n$ "tends to" the number 0. (Draw plot here…x-axis being n, y-axis a_n).

We can make this statement precise by writing

$$\lim a_n = \lim (1/2^n) = 0$$

(under the "lim", of course, you would write n→oo). This motivates the definition

Definition: A sequence {an} has the limit L and we write

$$\lim a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to oo$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If the lim a_n exists (is a number), we say the sequence converges. Otherwise, we say it diverges.

Ex. $a_n = (-1)^n$ is the sequence -1,1,-1,1,-1,1,….. Plotting it, we see that it bounces back between the two numbers -1 and 1, and thus diverges.

Ex. $a_n = (n+1)/n = 1 + 1/n$ goes 2, 3/2, 4/3, 5/4, …. We see that this sequences approaches the number 1. Thus, lim a_n = 1. We shall see soon how to rigorously justify this as well as be able to determine many similar limits.

Function Method for determining limits

Consider the previous example. Notice that if f(x) is the function

$$f(x) = (x+1)/x$$

we have $a_n = (n+1)/n = f(n)$. Since we know that $f(x) \rightarrow 1$ as $x \rightarrow oo$ (from previous chapters), it should be the case that a_n does as well. This example generalizes.

Theorem: If $\lim f(x) = L$ ($x \rightarrow oo$) and $a_n = f(n)$ (integer n), then

$$\lim a_n = L$$

Ex (very important example): Consider the function $f(x) = 1/x^r$
We know from way back that

$$\lim f(x) = \lim 1/x^r = 0 \qquad \text{when } r > 0$$

Thus, from the theorem, we know that

$$\lim f(n) = \lim 1/n^r = 0 \qquad \text{when } r > 0$$

So for example, wee see that $a_n = 1/\sqrt{n}$ goes to 0 as $n \rightarrow oo$ (take $r = ½$).

If the sequence a_n gets larger and larger as n gets bigger, then we reserve the notation:

$$\lim a_n = oo$$

and we say that a_n limits to infinity.

This happens, for example, with the sequence $a_n = n^2$.

Limit Laws

If we look at the sum of two sequences a_n, b_n , each having a limit, it seems intuitive that this new sequence $a_n + b_n$ should also have a limit, and its value should be the sum of the two limits. E.g. 1/n + (n+1)/n should go to 0 + 1 = 1.

This intuition is justified by the following limit laws.

Theorem: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$\lim (a_n + b_n) = \lim a_n + \lim b_n$

$\lim (a_n - b_n) = \lim a_n - \lim b_n$

$\lim (ca_n) = c \lim a_n$

$\lim (a_n b_n) = \lim a_n \lim b_n$

(*) $\lim (a_n / b_n) = \lim a_n / \lim b_n$ provided $\lim b_n$ not 0

$\lim (c) = c$

Ex: Consider the sequence $a_n = n^2/(2+n^2)$. This sequence seems to go to 1, but how to prove it. We can't simply apply rule (*) above directly since $\{n^2\}$ does not converge. However, we can **convert** the given sequence to look like:

$$a_n = 1/(2/n^2 + 1)$$

Now the numerator 1 and denominator $(2/n^2 + 1)$ do have finite limits. Thus, we see that $a_n \rightarrow 1/(0+1) = 1$.

In general, if one is given a rational function in n, this same technique applies.

Eg. Work out this technique for

$$a_n = (2n^2+1)/(n^2+n+1)$$

To verify that the limit here is 2.

Squeeze Method

There is also a squeeze theorem for sequences:

Theorem: If $a_n \le b_n \le c_n$ and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.

This theorem can be very useful.

Ex. Consider $a_n = (\sin n)/n$. We think that this sequence should go to 0. To prove it we should "squeeze" it between two other things that go to 0. Notice that

$$-1 \le (\sin n) \le 1 \qquad \text{for all } n.$$

Thus,

$$-1/n \le (\sin n)/n \le 1/n \qquad \text{for all } n.$$

Since $\lim (-1/n) = -\lim(1/n) = 0$, it follows that $a_n = (\sin n)/n$ does indeed go to 0.

The squeeze theorem also provides a proof of the following

Theorem: if $\lim |a_n| = 0$, then $\lim a_n = 0$.

Proof. We have $-|a_n| \le a_n \le |a_n|$. By assumption and the limit law with $c = -1$, the result follows from the squeeze theorem.

Ex. $a_n = (-1)^n/n \rightarrow 0$ since $1/n$ does.

L'Hostital's Rule Method

The idea here is to consider a sequence $a_n = f(n)$ coming from a function $f(x) = h(x)/g(x)$. From way back, we can sometimes compute a limit

$$\lim f(x) = \lim h(x)/g(x) = \lim h'(x)/g'(x)$$

using **L'Hostital's rule**.

Ex: Consider $a_n = (\ln n)/n$. What is the limit? First, from the **function method**, we can compute instead

$$\lim f(x) = \lim (\ln x)/x$$

since $f(x) = h(x)/g(x)$ for $h(x) = \ln x$ and $g(x) = x$, we compute

$$h'(x)/g'(x) = (1/x)/1 = 1/x$$

Taking the limit now gives us

$$\lim a_n = \lim (\ln x)/x = \lim (1/x) = 0.$$

A very important class of sequences are those of the form

$$a_n = r^n$$

where r is some fixed number. E.g. $(-1)^n$, 2^n, $1/2^n$, and so on.

The limit of such sequences is computed by the following theorem.

Geometric Sequence Theorem: If $-1 < r \leq 1$ then $\{r^n\}$ is convergent. Otherwise, it is divergent. When it converges, its value is

$$\lim r^n = \begin{cases} 0 & -1 < r < 1 \\ 1 & r = 1 \end{cases}$$

We now consider special types of sequences…Those that always get smaller or always get bigger.

Definition: The sequence $\{a_n\}$ is said to be increasing if $a_n < a_{n+1}$ for all n. It is said to be decreasing if $a_n > a_{n+1}$ for all n. It is said to be monotonic if it is either decreasing or increasing.

Ex. $a_n = 1/n$ is decreasing. $a_n = n-1$ is increasing. $a_n = 3/(n+5)$ is decreasing.

To check this, one verifies that $a_n < a_{n+1}$ for all n or that $a_n < a_{n+1}$ for all n

Ex. $a_n = 3/(n+5)$. Guess: decreasing. To verify, one looks at whether it is true that

(**) $$3/(n+5) > 3/(n+1+5)$$

so that we must have

$$n+6 > n+5$$

since this is **true** (working backwards) we see that the original inequality (**) holds and therefore, the sequence is decreasing.

Another method to determine whether a sequence is increasing or decreasing is to use the Function Method and verify that the derivative f'(x) is always positive or negative.

Ex. $a_n = n/(n^2+1)$. $f(x) = x/(x^2+1)$. $f'(x) = (1-x^2)/(x^2+1)^2 < 0$ whenever $x > 1$. Thus, the sequence is decreasing.

As we have seen, sequences that have limits are those that tend toward a number. Thus, those sequences are "bounded" in the sense that they can't get too big or too small. While a sequence might not have a limit, it can still be "bounded" in this sense. E.g. $(-1)^n$ doesn't have a limit, but it stays "bounded" between -1 and 1. This motivates the definition.

Definition: A sequence $\{a_n\}$ is bounded above if there is a number M such that

$$a_n \leq M \qquad \text{for all } n$$

It is bounded below if there is a number m such that

$$m \leq a_n \qquad \text{for all } n$$

A sequence $\{a_n\}$ is bounded if it is both bounded below and above.

Ex. $a_n = n$ is bounded below but not above. The sequence $a_n = (-1)^n$ is bounded both below and above and thus is bounded.

The main result relating bounded sequences and monotonic ones is

Monotonic Sequence Theorem: Every bounded sequence that is monotonic is convergent.

The basic idea here is that if a sequence keeps getting bigger (say) but cannot be any bigger than some number M, it must be the case that that sequence limits to something.

Ex. Before, we considered $a_n = n/(n^2+1)$ and showed it to be decreasing. One now checks that it is bounded above by 1

$$n/(n^2+1) < 1 \qquad ?$$

$$n < n^2 + 1 \qquad ?$$

This last inequality is true for $n \geq 1$; thus, a_n is bounded above. Clearly it is bounded below by 0. Therefore the monotonic sequence theorem shows that a_n has a limit.

Of course, we already can tell that this sequence has a limit using the techniques from before:

$$a_n = n/(n^2+1) = (1/n)/(1+1/n^2) \qquad \text{(dividing both top and bottom by } n^2\text{)}$$

and then using the limit law (*) to give:

$$\lim a_n = 0/(1+0) = 0.$$

Ex. Consider the example $a_1 = 2$, $a_{n+1} = (1/2)(a_n + 6)$.

This sequence goes like: 2, 4, 5, 5.5, 5.75, 5.875, 5.9375…

It seems to be increasing and bounded above 6. If we can prove these things, then the theorem above will tell us that $\{a_n\}$ has a limit. In fact, one can check these things using mathematical induction. One thus obtains that the limit of a_n as $n \to oo$ exists and we have $\lim a_n = L$. To compute this limit, we write:

$$L = \text{Lim } a_{n+1} = \lim (1/2)(a_n+6) = \lim a_n/2 + \lim 3 = (1/2) \lim a_n + 3 = (1/2)L + 3$$

Thus, solving for L, we find that $L = (1/2)L + 3$ so that indeed $L = 6$.