Worksheet on Proofs 2

Name:

Math 54 – Professor Craig Evans
TA: Chris Hillar

The purpose of this handout is to help with formalizing correct proofs (yes, again). Read *very carefully* the statements of the following problems before working on them.

Problem 1: (From last time) Let A be an $n \times n$ matrix **invertible** matrix. Prove:
(1) That 0 is **not** an eigenvalue of A.
(2) That if λ is an eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}.

Problem 2: Let $\underline{u}$ and $\underline{v}$ be vectors in $\mathbf{R}^n$ that are **not** perpendicular to each other. Prove that the two vectors

$$\underline{v} \quad \text{and} \quad \underline{u} - (\underline{u} \cdot \underline{v})\, \underline{v}$$

are perpendicular to each other **if and only if** the norm of $\underline{v}$ is equal to 1.

Problem 3: **(1)** Let $\underline{u}$ and $\underline{v}$ be nonzero vectors in $\mathbf{R}^n$ that are perpendicular to each other. Prove that they are linearly independent. **(2)** More generally, suppose that $\{\underline{u}_1,\ldots,\underline{u}_n\}$ is an orthonormal set of vectors in $\mathbf{R}^n$. Prove that they form a basis for $\mathbf{R}^n$.

Problem 4: Let A and B be $n \times n$ matrices and suppose that AB is invertible. Prove that both A and B must also be invertible.

Problem 5: Let A be an $n \times n$ matrix. Prove that the set of eigenvectors corresponding to an eigenvalue λ of the matrix A (along with the vector $\underline{0}$) is equal to NS($\lambda I - A$).

Problem 6: (Harder) **(1)** Let A be an $n \times n$ matrix and let $p(x)$ be the characteristic polynomial of A. Prove that $(-1)^n p(-x)$ is the characteristic polynomial for $-A$. **(2)** Suppose that $\{\lambda_1,\ldots,\lambda_n\}$ are the eigenvalues of A (listed with multiplicity). Use the first part to prove that $\{-\lambda_1,\ldots,-\lambda_n\}$ are all the eigenvalues of the matrix $-A$. (So, for example, if {2,2,0} are the eigenvalues of A, show that {-2,-2,0} are the eigenvalues of $-A$).

Problem 7: (Harder) Describe the set of **diagonalizable** 2×2 matrices A satisfying

$$A^2 = A$$

Problem 8: (Hardest) **(1)** Let A and B be $n \times n$ matrices and suppose that A is **invertible**. Let $p(x)$ be the characteristic polynomial of AB and let $q(x)$ be the characteristic polynomial of BA. Prove that $p(x) = q(x)$. **(2)** Using the first part of this problem, show that if either A or B is invertible, then AB and BA always have the same set of eigenvalues.
(Very difficult) **(3*)** Prove that even if both A and B are not invertible, part (1) is still true.