Vishik–Lyusternik's method and the inverse problem for plasma equilibrium in a tokamak

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Control over thermonuclear fusion reactions (including suppression of instabilities of the plasma discharge) depends essentially on how well the information about the current density through plasma is taken into account. In the case cylindrical approximation (when the tokamak (toroidal magnetic) chamber and the resulting plasma discharge are modeled in the form of infinite cylinders $\mathcal{S}\times\mathbb{R}$ and $\omega\times\mathbb{R}$ with simply connected cross-sections $\mathcal{S}\Subset\mathbb{R}^2$ and $\omega\Subset\mathcal{S}$), the required current distribution is given by the mapping $f_u : \omega \ni (x,y) \mapsto f\big(u(x,y)\big) \geq 0$, where the *required* functions $u \in C^2(\omega)$ and f are as follows:

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=f\big(u(x,y)\big) \quad \text{in} \quad \omega\,, \qquad \text{and} \qquad u=0 \quad \text{on} \quad \gamma=\partial\omega,$$

$$\sup_{P\in\gamma}\Big|\frac{\partial u}{\partial \nu}(P)-\ \Phi(P)\Big|\leq \lambda \sup_{P\in\gamma}\Big|\Phi(P)\Big|\,, \qquad \int_\gamma \frac{\partial u}{\partial\nu}\,d\gamma=1=\text{the total current}\,.$$

Here, $\gamma=\overline{\omega}\setminus\omega$ is the boundary of the domain ω, $\lambda\geq 0$ is small parameter, ν is the outward unit normal to the curve $\gamma=\partial\omega$ (with respect to the domain ω). Both the function Φ and the curve $\gamma=\partial\omega$ (and hence the domain ω) may be regarded as known: they are determinable from measurements of the magnetic field at the tokamak chamber $\partial\mathcal{S}$.

Within the class of affine functions $f : u \mapsto f(u)=au+b$, Vishik–Lyusternik's method is capable of showing that

$$\left|\frac{\partial u}{\partial\nu}\Big|_{s\in\gamma}-\left(\frac{1}{|\gamma|}-\frac{k(s)-|\gamma|^{-1}\int_\gamma k(s)\,ds}{2|\gamma|\sqrt{a}}\right)\right|\leq\frac{C_\gamma(a)}{\sqrt{a}}\,,\ C_\gamma(a)\to 0\ \text{ as }\ a\to\infty\,,$$

$$\left|\frac{d}{da}\frac{\partial u}{\partial\nu}\Big|_{s\in\gamma}-\frac{k(s)-|\gamma|^{-1}\int_\gamma k(s)\,ds}{4|\gamma|\,a^{3/2}}\right|\leq\frac{C_\gamma(a)}{a^{3/2}}\,, \quad C_\gamma(a)\to 0 \quad \text{as} \quad a\to\infty\,,$$

where $k(s)$ is the curvature of γ at $s\in\gamma$. Using these asymptotic relations, it follows that there is only one affine distribution f_u (for a large class of domains ω) if $\lambda=0$. However, for any arbitrarily small $\lambda>0$, there is an infinite number $\{f_u^j\}_{j\in\mathbb{N}}$ of distributions, for which $\|f_u^{j_1}\|\ll\|f_u^{j_2}\|$, $j_1\neq j_2$, where $\|f_u^j\|=\max\limits_{(x,y)\in\omega}\big|f_u^j(x,y)\big|$. It is shown that all these different distributions are necessarily members of a sequence converging to the δ-function supported on γ (the so-called skinned current). Hence these distributions are not essentially different from the physical standpoint.

Two truly physically essentially different current distributions f_u^1 and f_u^2 are found in the class of polynomials $f : u\mapsto f(u)=\sum_{m=0}^{3} a_m u^m$ of third degree (see Russian J. Math. Physics, **17** (1), 56–65 (2010) and Asymptotic Analysis, **74** (1), 95–121 (2011)).