Negative eigenvalues of two-dimensional Schrödinger operators

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Given a non-negative L^1_{loc} function $V(x)$ on $\mathbb{R}^n$, consider the Schrödinger operator $H_V = -\Delta - V$ where $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator. More precisely, H_V is defined as a form sum of $-\Delta$ and $-V$, so that, under certain assumptions about V, the operator H_V is self-adjoint in $L^2(\mathbb{R}^n)$.

Denote by $\operatorname{Neg}(V)$ the number of non-positive eigenvalues of H_V (counted with multiplicity), assuming that its spectrum in $(-\infty, 0]$ is discrete. For example, the latter is the case when $V(x) \to 0$ as $x \to \infty$. We are are interested in obtaining estimates of $\operatorname{Neg}(V)$ in terms of the potential V in the case $n = 2$.

For the operator H_V in $\mathbb{R}^n$ with $n \geq 3$ a celebrated inequality of Cwikel-Lieb-Rozenblum says that

$$\operatorname{Neg}(V) \leq C_n \int_{\mathbb{R}^n} V(x)^{n/2}\, dx. \tag{1}$$

For $n = 2$ this inequality is not valid. Moreover, no weighted L^1-norm of V can provide an upper bound for $\operatorname{Neg}(V)$. In fact, in the case $n = 2$ instead of the upper bounds, the lower bound in (1) is true.

The main result is the estimate (2) below that was obtained jointly with N.Nadirashvili. For any $n \in \mathbb{Z}$, set

$$U_n = \begin{cases} \{e^{2^{n-1}} < |x| < e^{2^n}\}, & n > 0, \\ \{e^{-1} < |x| < e\}, & n = 0, \\ \{e^{-2^{|n|}} < |x| < e^{-2^{|n|-1}}\}, & n < 0. \end{cases}$$

Define for any $n \in \mathbb{Z}$ the following quantities:

$$A_n = \int_{U_n} V(x)\,(1 + |\ln|x||)\, dx\ , \qquad B_n = \Big(\int_{\{e^n < |x| < e^{n+1}\}} V^p(x)\, |x|^{2(p-1)}\, dx \Big)^{1/p},$$

where $p > 1$ is fixed. Then the following estimate holds

$$\operatorname{Neg}(V) \leq 1 + C \sum_{\{n \in \mathbb{Z}: A_n > c\}} \sqrt{A_n} + C \sum_{\{n \in \mathbb{Z}: B_n > c\}} B_n, \tag{2}$$

where C, c are positive constants depending only on p.

For example, (2) implies the finiteness of $\operatorname{Neg}(V)$ provided V is locally bounded and $V(x) = o\Big(\frac{1}{|x|^2 \ln^2 |x|}\Big)$ as $x \to \infty$, which cannot be seen by any previously known method.