Sharp two-term Sobolev inequality and applications to the Lieb–Thirring estimates

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For a function $\varphi \in H^1(\mathbb{R})$ the following inequality is well known

$$\|\varphi\|_\infty^2 \leq \|\varphi\|\|\varphi'\|,$$

where the norms on the right-hand side are the L_2-norms. The constant 1 in the inequality is sharp and the unique extremal unction is $\varphi^*(x) = e^{-|x|}$. The same inequality clearly holds on a finite interval, that is, for $\varphi \in H_0^1(0, L)$. However, since $\varphi^*(x) > 0$, no extremal functions exist. The following result provides a sharp correction term for this inequality (the correction term in the periodic case was found in [1]).

Theorem 1. *Let $f \in H_0^1(0, L)$. Then* $\|\varphi\|_\infty^2 \leq \|\varphi\|\|\varphi'\|\bigl(1 - 2e^{-\frac{L\|\varphi'\|}{\|\varphi\|}}\bigr)$. *The coefficients of the two terms on the right-hand side are sharp and no extremal functions exist.*

As in the periodic case considered in [2], this theorem makes it possible to obtain a simultaneous bound for the negative trace and the number of negative eigenvalues for the 1D Schrödinger eigenvalue problem on $(0, L)$

$$-y_j'' - Vy_j = \nu_j y_j \tag{1}$$

with Dirichlet boundary conditions $y(0) = y(L) = 0$ and potential $V(x) \geq 0$.

Theorem 2. *Suppose that there exist N negative eigenvalues $\nu_j \leq 0$, $j = 1, \dots, N$ of the operator* (1). *Then both the negative trace and the number N of negative eigenvalues satisfy for any $\varepsilon \geq 0$*

$$\sum_{j=1}^N |\nu_j| + N \cdot \frac{\pi^2}{L^2}\left(\frac{c(\varepsilon)}{1+\varepsilon}\right)^2 \leq \frac{2}{3\sqrt{3}} \cdot (1+\varepsilon) \int_0^L V(x)^{3/2} dx,$$

where $c(\varepsilon) = \min_{x\geq 1}\bigl(\varepsilon x + 2xe^{-\pi x}\bigr)$.

Remark. We have $c(\varepsilon) \geq \varepsilon$, and the optimal ε for the bound involving only the number of negative eigenvalues N is $\varepsilon = 2$.

References

[1] M.V. Bartuccelli, J. Deane, and S.V. Zelik, *Asymptotic expansions and extremals for the critical Sobolev and Gagliardo-Nirenberg inequalities on a torus.* arXiv:1012.2061 (2010).

[2] A.A.Ilyin, *Lieb–Thirring inequalities on some manifolds*, Journal of Spectral Theory **2**:1 (2012), 1–22.