On the 2-point problem for the Lagrange-Euler equation

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Consider the motion of ideal incompressible fluid in a bounded domain (or on a compact Riemannian manifold). The configuration space of the fluid is the group of volume preserving diffeomorphisms of the flow domain, and the flows are geodesics on this infinite-dimensional group where the metric is defined by the kinetic energy. The geodesic equation is the Lagrange-Euler equation. The problem usually studied is the initial value problem, where we look for a geodesic with given initial fluid configuration and initial velocity field. In this talk we consider a different problem: find a geodesic connecting two given fluid configurations. The main result is the following

Theorem: *Suppose the flow domain is a 2-dimensional torus. Then for any two fluid configurations there exists a geodesic connecting them. This means that, given arbitrary fluid configuration (diffeomorphism), we can "push" the fluid along some initial velocity field, so that by time one the fluid, moving according to the Lagrange-Euler equation, assumes the given configuration.*

This theorem looks superficially like the Hopf-Rinow theorem for finite-dimensional Riemannian manifolds. In fact, these two theorems have almost nothing in common. In our case, unlike the Hopf-Rinow theorem, the geodesic is not, in general case, the shortest curve connecting the endpoints (fluid configurations). Moreover, the length minimizing curve can not exist at all, while the geodesic always exists.

The proof is based on some ideas of global analysis (Fredholm quasilinear maps) and microlocal analysis of the Lagrange-Euler equation (which may be called a "microglobal analysis").