Homogenization of the elliptic Dirichlet problem: operator error estimates

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Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O};\mathbb{C}^n)$, we consider a matrix elliptic differential operator $A_\varepsilon = b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$ with the Dirichlet boundary condition. We assume that an $(m \times m)$-matrix-valued function $g(\mathbf{x})$ is bounded, uniformly positive definite and periodic with respect to some lattice Γ. The elementary cell of Γ is denoted by Ω. Next, $b(\mathbf{D}) = \sum_{j=1}^d b_j D_j$ is an $(m \times n)$-matrix first order differential operator (b_j are constant matrices). It is assumed that $m \geq n$ and the symbol $b(\boldsymbol{\xi}) = \sum_{j=1}^d b_j \xi_j$ has maximal rank, i. e., $\operatorname{rank} b(\boldsymbol{\xi}) = n$ for $0 \neq \boldsymbol{\xi} \in \mathbb{R}^d$. The simplest example is $A_\varepsilon = -\operatorname{div} g(\mathbf{x}/\varepsilon)\nabla$.

We study the behavior of the solution $\mathbf{u}_\varepsilon$ of the Dirichlet problem $A_\varepsilon \mathbf{u}_\varepsilon = \mathbf{F}$ in $\mathcal{O}$, $\mathbf{u}_\varepsilon|_{\partial\mathcal{O}} = 0$, where $\mathbf{F} \in L_2(\mathcal{O};\mathbb{C}^n)$. It turns out that $\mathbf{u}_\varepsilon$ converges in $L_2(\mathcal{O};\mathbb{C}^n)$ to $\mathbf{u}_0$, as $\varepsilon \to 0$. Here $\mathbf{u}_0$ is the solution of the "homogenized" Dirichlet problem $A^0 \mathbf{u}_0 = \mathbf{F}$ in $\mathcal{O}$, $\mathbf{u}_0|_{\partial\mathcal{O}} = 0$. The *effective operator* A^0 is given by the expression $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ with the Dirichlet boundary condition. The effective matrix g^0 is a constant positive $(m \times m)$-matrix defined as follows. Denote by $\Lambda(\mathbf{x})$ the $(n \times m)$-matrix-valued periodic solution of the equation $b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0$ such that $\int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0$. Then $g^0 = |\Omega|^{-1} \int_\Omega g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m)\, d\mathbf{x}$.

Theorem 1. (see [2]) *We have the following sharp order error estimate*:

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{L_2(\mathcal{O};\mathbb{C}^n)} \leq C\varepsilon \|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}.$$

Now we give approximation of $\mathbf{u}_\varepsilon$ in the Sobolev space $H^1(\mathcal{O};\mathbb{C}^n)$. For this, the first order corrector must be taken into account.

Theorem 2. (see [1]) 1) *Let $\Lambda \in L_\infty$, and denote $\Lambda^\varepsilon(\mathbf{x}) = \Lambda(\varepsilon^{-1}\mathbf{x})$. Then*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon \Lambda^\varepsilon b(\mathbf{D})\mathbf{u}_0\|_{H^1(\mathcal{O};\mathbb{C}^n)} \leq C\varepsilon^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}.$$

2) *In the general case, we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0 - \varepsilon \Lambda^\varepsilon b(\mathbf{D})(S_\varepsilon \widetilde{\mathbf{u}}_0)\|_{H^1(\mathcal{O};\mathbb{C}^n)} \leq C\varepsilon^{1/2} \|\mathbf{F}\|_{L_2(\mathcal{O};\mathbb{C}^n)}.$$

Here $\widetilde{\mathbf{u}}_0 = P_\mathcal{O} \mathbf{u}_0$ and $P_\mathcal{O} : H^2(\mathcal{O};\mathbb{C}^n) \to H^2(\mathbb{R}^d;\mathbb{C}^n)$ is a continuous extension operator, S_ε is the smoothing operator $(S_\varepsilon \mathbf{u})(\mathbf{x}) = |\Omega|^{-1} \int_\Omega \mathbf{u}(\mathbf{x} - \varepsilon\mathbf{z})\, d\mathbf{z}$.

References

[1] M. A. Pakhnin, T. A. Suslina, *Operator error estimates for homogenization of the Dirichlet problem in a bounded domain*, Preprint, 2012. Available at http://arxiv.org/abs/1201.2140.

[2] T. A. Suslina, *Homogenization of the elliptic Dirichlet problem: operator error estimates in L_2*, Preprint, 2012. Available at http://arxiv.org/abs/1201.2286.