Algebra of boundary value problems with small parameter

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In a singular perturbation problem one is concerned with a differential equation of the form $A(\varepsilon)u_\varepsilon = f_\varepsilon$ with initial or boundary conditions $B(\varepsilon)u_\varepsilon = g_\varepsilon$, where ε is a small parameter. The distinguishing feature of this problem is that the orders of $A(\varepsilon)$ and $B(\varepsilon)$ for $\varepsilon \neq 0$ are higher than the orders of $A(0)$ and $B(0)$, respectively. There is by now a vast amount of literature on singular perturbation problems for partial differential equations. A comprehensive theory of such problems was initiated by the remarkable paper of Vishik and Lyusternik [1]. In [2], Volevich completed the theory of differential boundary value problems with small parameter by formulating the Shapiro-Lopatinskii type ellipticity condition.

We contribute to the theory by constructing an algebra of pseudodifferential operators in which singularly perturbed boundary value problems can be treated. Given any $m, \mu \in \mathbb{R}$, denote by $\mathcal{S}^{m,\mu}$ the space of all smooth functions $a(x, \xi, \varepsilon)$ on $T^*\mathbb{R}^n \times \mathbb{R}_{\geq 0}$, such that $|D_x^\alpha D_\xi^\beta a| \leq C_{\alpha,\beta} < \xi >^{\mu-|\beta|} < \varepsilon\xi >^{m-\mu}$ for all multi-indices α and β, where $C_{\alpha,\beta}$ are constants independent of x, ξ and ε. For any fixed $\varepsilon > 0$, a function $a \in \mathcal{S}^{m,\mu}$ is a symbol of order m on $\mathbb{R}^n$ which obviously degenerates as $\varepsilon \to 0$. These symbols quantize to continuous operators $H^{r,s} \to H^{r-m,s-\mu}$ in a scale of Sobolev spaces on $\mathbb{R}^n$ whose norms depend on ε and are based on L^2 and weight functions $< \xi >^s < \varepsilon\xi >^{r-s}$. The family $\mathcal{S}^{m-j,\mu-j}$ with $j = 0, 1, \dots$ is used as usual to define asymptotic sums of homogeneous symbols. By the homogeneity of degree μ is meant the property $a(x, \lambda\xi, \lambda^{-1}\varepsilon) = \lambda^\mu a(x, \xi, \varepsilon)$ for all $\lambda > 0$. Let $\mathcal{S}^{m,\mu}_{\mathrm{phg}}$ stand for the subspace of $\mathcal{S}^{m,\mu}$ consisting of all polyhomogeneous symbols, i.e., those admitting asymptotic expansions in homogeneous symbols. For any $a \in \mathcal{S}^{m,\mu}_{\mathrm{phg}}$ there is well-defined principal homogeneous symbol $\sigma^\mu(a)$ of degree μ whose invertibility away from the zero section of $T^*\mathbb{R}^n$ is said to be the interior ellipticity with small parameter. Familiar techniques lead now to calculi of pseudodifferential operators with small parameter on diverse compactifications of smooth manifolds. Our results gain in interest if we realize that pseudodifferential operators with small parameter provide also adequate tools for studying Cauchy problems for elliptic equations.

This is a joint paper with my PhD student Evgeniya Dyachenko who studies singular perturbation problems.

References

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