Example of equations with nonlinearity of type min[u,v]

N. Vvedenskaya
Institute for Information Transmission Problems, Moscow, Russia
ndv@iitp.ru

Y.M. Suhov
IITP, Cambridge University, Universidade de Sao Paulo
yms@statslab.cam.ac.uk

This paper considers a boundary-value problem for a nonlinear system of equations that are derived from a trading process model [1].

Let $u(x.t), v(x,t)$, $0 < x < 1$, $t > 0$, satisfy a system

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= -a_u \frac{\partial u(x,t)}{\partial x} \quad -b_u u(x,t) - c\min[u(x,t), v(x,t)], \\
\frac{\partial v(x,t)}{\partial t} &= \quad a_v \frac{\partial v(x,t)}{\partial x} \quad -b_v v(x,t) - c\min[u(x,t), v(x,t)],
\end{aligned} \tag{1}$$

$$\frac{\partial u(0,t)}{\partial t} = d_u - a_u u(0,t) - b_u u(0,t) - c\min[u(0,t), v(0,t)], \tag{2}$$

$$\frac{\partial v(1,t)}{\partial t} = d_v - a_v v(1,t) - b_v v(1,t) - c\min[u(1,t), v(1,t)], \tag{3}$$

$$u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0. \tag{4}$$

Here $a_{u/v}$, $b_{u/v}$, c, $d_{u/v}$ are positive.

Theorem 1 *For any initial data $u_0(x), v_0(x)$ for all $t > 0$ there exists a unique solution to (*1)-(4). As $t \to \infty$ a solution approaches a fixed point which is the unique solution to a system

$$\begin{aligned}
-a_u \frac{\mathrm{d}u(x,t)}{\mathrm{d}x} \quad -b_u u(x,t) - c\min[u(x,t), v(x,t)] = 0, \\
a_v \frac{\mathrm{d}v(x,t)}{\mathrm{d}x} \quad -b_v v(x,t) - c\min[u(x,t), v(x,t)] = 0,
\end{aligned} \tag{5}$$

$$d_u - a_u u(0,t) - b_u u(0,t) - c\min[u(0,t), v(0,t)] = 0, \tag{6}$$

$$d_v - a_v v(1,t) - b_v v(1,t) - c\min[u(1,t), v(1,t)] = 0. \tag{7}$$

Remark that the type of boundary conditions (2), (3) and (6), (7) depends on sign of $u - v$, that makes not evident the uniqueness of solution to (5)-(7).

The first author was supported by grant RFBR 11-01-00485a.

References

[1] N.D. Vvedenskaya, Y. Suhov, V. Belitsky, *A non-linear model of limit order book dynamics*, http://arXiv:1102.1104 (2011).