INTEGRAL OPERATORS WITH TWO-SIDED CUSP SINGULARITIES

ANDREW COMECH AND SCIPIO CUCCAGNA

June 19, 2000

ABSTRACT. We consider the Fourier integral operators associated to singular canonical relations, with the cusp singularities on both sides. We prove that such operators lose $\frac{1}{4}+\epsilon$ of a derivative in smoothing properties, compared to non-singular Fourier integral operators. We also state the results on regularity properties in L^p spaces. Our approach is based on almost orthogonality decompositions of singular oscillatory integral operators.

INTRODUCTION

In this paper, we continue the study of the relation of properties of Fourier integral operators $\mathfrak{F} : C_{\mathrm{comp}}(Y) \to C'(X)$ to the geometry of the projections $\mathcal{C} \to T^*X$ and $\mathcal{C} \to T^*Y$ from the canonical relation $\mathcal{C} \subset T^*X\backslash 0 \times T^*Y\backslash 0$ associated to $\mathfrak{F}$. The standard theory developed by Hörmander [Hö 71] is applicable to operators associated to canonical relations which are locally graphs of symplectomorphisms from T^*X to T^*Y (when π_L and π_R are locally diffeomorphisms). On the other hand, in a number of natural cases such as scattering theory [MT 85], generalized Radon transforms (see the surveys [Ph 94] and [GrSeW 97]), trace operators [T 98], etc., the projections π_L and π_R have singular points. The case when π_L and π_R have Whitney fold singularities was studied in [MT 85]. One-sided fold singularities appeared in [GrU 91]. In the beginning of the 90's Phong and Stein [PhSt 91] gave a new impulse to the topic, proposing a program of studies of properties of singular operators. Stimulated by this work, various authors considered a number of cases. Let us mention the papers [PhSt 97], [Se 98], and [Ry 99] where the optimal results for the case $\dim X = \dim Y = 1$ were obtained. For the results in higher dimension, see the survey [GrSeW 97].

In this paper we consider the case when both π_L and π_R have cusp singularities, that is, in certain local coordinates have the form [Mo 65]

$$(1.1)\qquad (x_1, \ldots, x_m) \mapsto (y_i = x_i \text{ for } i = 1 \ldots m-1,\ y_m = x_m^3 - x_1 x_m).$$

Both authors were partially supported by grants from the National Science Foundation

This case includes Radon transforms associated to generic families of curves in $\mathbb{R}^4$, see [Proposition 2.2, GrSeW 97]. Below, we will prove the following result:

Theorem 1.1. *Let* $\dim X = \dim Y = n$ *and let* $\mathfrak{F} \in I^\mu(X, Y, \mathcal{C})$ *be a Fourier integral operator of order* μ*, associated to the canonical relation* $\mathcal{C}$*. Assume that both projections* $\pi_{_L} : \mathcal{C} \to T^*X$ *and* $\pi_{_R} : \mathcal{C} \to T^*Y$ *have at most cusp singularities. Then* $\mathfrak{F}$ *extends to a continuous operator from* $L^2_{\alpha,\mathrm{comp}}(Y)$ *to* $L^2_{\beta,\mathrm{loc}}(X)$ *if* $\mu < \alpha - \beta - \frac{1}{4}$*.*

Further, set $w_{_R} = 1$ *if at the cusp points* $\pi_{_R}$ *is a strong cusp (see Definition 1.3 below) and* $w_{_R} = 2$ *otherwise; similarly set* $w_{_L} = 1$ *if at the cusp points* $\pi_{_L}$ *is a strong cusp and* $w_{_L} = 2$ *otherwise. Additionally, assume that* $\mathcal{C} \to X$ *and* $\mathcal{C} \to Y$ *are submersions. Then, for* $1 < p < \frac{w_R+2}{w_R+1}$ *and* $2 + w_{_L} < p < \infty$*,* $\mathfrak{F}$ *extends to a continuous operator from* $L^p_{\alpha,\mathrm{comp}}(Y)$ *to* $L^p_{\beta,\mathrm{loc}}(X)$ *if* $\mu < \alpha - \beta - \alpha_p$*, with* $\alpha_p = (n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$*. The* L^p*-continuity of* $\mathfrak{F}$ *for* p *between* $\frac{w_R+2}{w_R+1}$ *and* $2 + w_{_L}$ *is obtained by interpolation with the* L^2 *estimates.*

Let us summarize previously known results on Sobolev continuity of singular Fourier integral operators in higher dimensions (the continuity properties of such operators are usually expressed in terms of the loss of derivatives versus the continuity of operators associated to local graphs). Known results include the loss of $\frac{1}{6}$ of a derivative in the case of canonical relations with two-sided Whitney folds [MT 85], the loss of at most $\frac{1}{4}$ of a derivative in the case of the Whitney fold on either side and with no assumption on the other projection [GrSe 94], and more accurately the loss of $1/(4 + 2k^{-1})$ of a derivative in the case of the Whitney fold on one side and a singularity of type k on the other side [Co 99].

For operators with higher order singularities, Greenleaf and Seeger proved that there is a loss of at most $\frac{1}{3}$ of a derivative when one of the projections from the canonical relation has a cusp singularity [GrSe 98].

Theorem 1.1 gives nearly the optimal estimate, the loss of $\frac{1}{4} + \epsilon$ of a derivative, in the case when both projections are cusps. We expect that there is a loss of exactly $\frac{1}{4}$, which would be the optimal result, but we can not prove this yet.

The $L^p \to L^p$ estimates on classical Fourier integral operators were obtained in [SeSoSt 91]: *A Fourier integral operator* $\mathfrak{F} \in I^\mu(X, Y, \mathcal{C})$ *of order* μ *associated to a canonical relation which is a local graph is continuous from* $L^p_{\alpha,\mathrm{comp}}(Y)$ *to* $L^p_{\beta,\mathrm{loc}}(X)$*,* $1 < p < \infty$*, if* $\mu \le \alpha - \beta - \alpha_p$ *where* $\alpha_p = (n-1)|\frac{1}{p} - \frac{1}{2}|$. The operators with two-sided fold singularities were considered in [SmSo 94], and the operators with the fold singularity on one side and higher order singularity on the other were considered in [CoCu 98]. The basic result is that for p away from some neighborhood around $p = 2$, the $L^p \to L^p$ continuity of singular Fourier integral operators is the same as that of operators associated to local graphs. The L^p continuity of operators with two-sided cusp singularities stated in Theorem 1.1 seems almost optimal (we expect that there is no loss of ϵ anywhere except for the points $p = \frac{w_R+2}{w_R+1}$ and $p = 2 + w_{_L}$).

We will characterize the properties of π_L and π_R in terms of the type of singularity at a point. Let M and N be two smooth manifolds of the same dimension and let $\pi : M \to N$ be a smooth map. Let $\Sigma = \{p \in M \mid \det d\pi|_p = 0\}$ be the critical variety of the map π ($\det d\pi$ denotes the determinant of the Jacobi matrix of π in certain local coordinates). Assume that π drops rank simply by 1:

$$\dim \operatorname{Ker} d\pi \le 1, \qquad d(\det d\pi)|_\Sigma \ne 0.$$

Definition 1.2. *The type of the map π at a critical point $p \in \Sigma$ is the smallest integer k such that $V^k(\det d\pi)|_p \ne 0$, where V is a smooth vector field over M which generates* $\operatorname{Ker} d\pi$*:*

$$V|_\Sigma \in \operatorname{Ker} d\pi, \qquad V|_\Sigma \ne 0.$$

The definition does not depend on the choice of V.

Type 1 corresponds to the maps with Whitney fold singularities ($S_{1,0}$ Morin singularities). The equivalence of the maps of type 2 and cusp singularities ($S_{1,1,0}$ Morin singularities) easily follows from [Mo 65]. In particular, the map (1.1) is of type 2 in the origin since the determinant of the mixed Hessian is $\det d\pi(x) = 3x_m^2 - x_1$, while the kernel of $d\pi$ is generated by $V = \partial_{x_m}$. (For $k > 2$, the maps with Morin singularity $S_{1_k,0}$ at certain points are also of type k at those points, although the converse is not necessarily true.)

For the projections from the canonical relation $\mathcal{C} \subset T^*X \times T^*Y$, we also define a strong cusp, similarly to [GrSe 98]:

Definition 1.3. *We say that the map $\pi : \mathcal{C} \to T^*X$ has a strong cusp singularity at a point $p \in \mathcal{C}$ if π has a cusp singularity at that point, and additionally*

$$d(\det d\pi)|_{\operatorname{Ker} d(\pi_X \circ \pi)|_p} \ne 0.$$

Above, π_X stands for the canonical projection $T^*X \to X$.

Remark. The definition of cusp (1.1) already implies that $d(\det d\pi) \ne 0$, but in the above definition we additionally require that $d(\det d\pi)$ takes non-zero values on "directions orthogonal to X". A weaker condition, the type of π relative to $\operatorname{Ker}(d\pi_X \circ d\pi)$, is introduced in [CoCu 98]. If π is a cusp, its type relative to $\operatorname{Ker}(d\pi_X \circ d\pi)$ could only be equal to 1 (for a strong cusp) or to 2.

It is well known [GrSe 94] that the L^2 theory of Fourier integral operators reduces to the analysis of oscillatory integral operators of the form

$$T_\lambda u(x) = \int_{\mathbb{R}^n} e^{i\lambda S(x,\vartheta)} a(x,\vartheta) u(\vartheta) d\vartheta, \tag{1.2}$$

where $S \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $a \in C^\infty_{\text{comp}}(\mathbb{R}^n \times \mathbb{R}^n)$, and $\lambda \gg 1$. The associated canonical relation $\mathcal{C} = \{(x, d_x S), (\vartheta, d_\vartheta S)\} \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n$, and the projections from $\mathcal{C}$ can be represented by

$$\pi_{{}_L} : \ (x,\vartheta) \mapsto (x, S_x(x,\vartheta)), \qquad \pi_{{}_R} : \ (x,\vartheta) \mapsto (\vartheta, S_\vartheta(x,\vartheta)).$$

The determinants of the Jacobi matrices of $\pi_{{}_L}$ and $\pi_{{}_R}$ are both equal to the determinant of the mixed Hessian of S, which we will denote by h:

$$h(x,\vartheta) = \det S_{x\vartheta}(x,\vartheta) = \det_{i,j} \frac{\partial^2 S(x,\vartheta)}{\partial x_i \, \partial \vartheta_j}.$$

This shows that the projections $\pi_{{}_L}$ and $\pi_{{}_R}$ have a common critical variety

$$\Sigma = \{(x,\vartheta) \mid h(x,\vartheta) = 0\},$$

although their behavior at critical points could be quite different.

We assume that both $\pi_{{}_L}$ and $\pi_{{}_R}$ have cusp singularities. The condition that e.g. $\pi_{{}_L}$ is a strong cusp at some point (x_o, ϑ_o) is equivalent to the requirements that $\pi_{{}_L} : (x,\vartheta) \mapsto (x, S_x(x,\vartheta))$ is a cusp and that $d_\vartheta(\det d\pi_{{}_L}) \neq 0$. (It then follows that the restriction $\pi_{{}_L}|_{x_o} : \ \vartheta \mapsto S_x(x_o,\vartheta)$ is a cusp at ϑ_o.) Let us emphasize that it is due to the inequalities $d_\vartheta(\det d\pi_{{}_L}) \neq 0$ and $d_x(\det d\pi_{{}_R}) \neq 0$ that one can derive slightly improved L^p estimates for canonical relations with strong cusps. See [CoCu 98] for details.

According to [GrSe 94], the L^2 part of Theorem 1.1 follows from the following result:

Theorem 1.4 (Main Theorem). *For any* $\epsilon > 0$, $\quad \|T_\lambda\|_{L^2 \to L^2} \leq c_\epsilon \lambda^{-\frac{n}{2}+\frac{1}{4}+\epsilon}$.

We prove this Theorem in Sections 2 and 3. The proof elaborates the ideas from [PhSt 91] and is based on the Cotlar-Stein Lemma. In particular, the present proof is directly related to the almost orthogonality argument in [Co 99].

In Section 4, we show how to obtain the L^p estimates of Theorem 1.1 via more precise formulation of L^2 estimates and interpolation with $\mathrm{H}^1 \to L^1$ estimates. The argument is basically the same as in [CoCu 98], where the operators with one-sided fold singularities were considered.

2. Proof of the Main Theorem

It suffices to prove the Main Theorem in the case when the density $a(x,\vartheta)$ in (1.2) is supported in a small neighborhood of a point (x_o,ϑ_o) where $\pi_{{}_L}$ and $\pi_{{}_R}$ are of type 2. We choose local coordinates $x = (x', x_n)$, $\vartheta = (\vartheta', \vartheta_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ so that

$$\text{(2.1)} \qquad \operatorname{rank} S_{x'\vartheta'} = n-1, \qquad \partial_{x_n}|_{(x_o,\vartheta_o)} \in \operatorname{Ker} d\pi_{{}_R}, \qquad \partial_{\vartheta_n}|_{(x_o,\vartheta_o)} \in \operatorname{Ker} d\pi_{{}_L}.$$

As in [Co 99], we introduce the vector fields

$$(2.2)\quad \mathcal{R} = \partial_{x_n} - S_{x_n\vartheta'}(x,\vartheta)S^{\vartheta' x'}(x,\vartheta)\partial_{x'}\,, \quad \mathcal{L} = \partial_{\vartheta_n} - S_{x'\vartheta_n}(x,\vartheta)S^{\vartheta' x'}(x,\vartheta)\partial_{\vartheta'},$$

where $S^{\vartheta' x'}(x,\vartheta)$ is the matrix inverse to $S_{x'\vartheta'}(x,\vartheta)$. $\mathcal{L}$ and $\mathcal{R}$ generate the kernels of of $d\pi_L$ and $d\pi_R$, respectively. Due to the choice of local coordinates, $|S_{x\vartheta_n}(x_o,\vartheta_o)| = |S_{x_n\vartheta}(x_o,\vartheta_o)| = 0$. Therefore, for any small constant $\varepsilon > 0$ (for the definiteness we assume that $\varepsilon < \frac{1}{2}$), we can localize $a(x,\vartheta)$ to a sufficiently small neighborhood of (x_o,ϑ_o) so that on the support of $a(x,\vartheta)$

$$(2.3)\qquad |S_{x\vartheta_n}(x,\vartheta)| + |S_{x_n\vartheta}(x,\vartheta)| \le \varepsilon.$$

Since both $d\pi_L$ and $d\pi_R$ are of type 2 at (x_o,ϑ_o), we may assume that everywhere on the support of $a(x,\vartheta)$

$$(2.4)\qquad \mathcal{L}^2 h(x,\vartheta) \neq 0, \qquad \mathcal{R}^2 h(x,\vartheta) \neq 0.$$

We need a dyadic partition of unity in $\mathbb{R}$. We pick a smooth non-negative symmetric function β_0 with $\operatorname{supp}\beta_0 \subset [-1,1]$, and $\beta_0(t) \equiv 1$ for $|t| \le \frac{1}{2}$. Define $\beta \in C^\infty_{\rm comp}([\frac{1}{2},2])$ by $\beta(t) = \beta_0(t/2) - \beta_0(t)$ for $t \ge 0$. For $j \in \mathbb{N}$, we define $\beta_j(t) = \beta(t/2^{j-1})$ and $\beta_{-j}(t) = \beta(-t/2^{|j|-1})$. There is the following dyadic partition of unity: $\sum\limits_{j\in\mathbb{Z}} \beta_j(t) \equiv 1$ for any $t \in \mathbb{R}$.

We decompose the operator (1.2) into $T_\lambda = \sum_{j,k\in\mathbb{Z}} T_\lambda^{j\,k}$, and further

$$T_\lambda^{j\,k} = \sum_{\pm} \sum_{\substack{\hbar > \hbar_o(\lambda,j,k)\\ \hbar = 2^{-N},\ N\in\mathbb{Z}}} T_{\lambda,\pm}^{\hbar\,j\,k} + T_{\lambda,0}^{\hbar_o(\lambda,j,k)\,j\,k},$$

where $\hbar_o = 2^{-N_o}$ with $N_o(\lambda,j,k) = [\log_2(\lambda^{1/2-\epsilon}2^{-\frac{|j|+|k|}{2}})]$, $T_{\lambda,\pm}^{\hbar\,j\,k}$ is defined by

$$T_{\lambda,\pm}^{\hbar\,j\,k} u(x) = \int\limits_{\mathbb{R}^n} e^{i\lambda S(x,\vartheta)} a(x,\vartheta)\beta(\frac{h(x,\vartheta)}{\pm\hbar})\beta_j(\frac{\mathcal{R}h(x,\vartheta)}{\hbar^{1/2}})\beta_k(\frac{\mathcal{L}h(x,\vartheta)}{\hbar^{1/2}})u(\vartheta)d\vartheta,$$

and $T_{\lambda,0}^{\hbar\,j\,k}$ is given by the same expression with $\beta_0(\frac{h(x,\vartheta)}{\hbar})$ instead of $\beta(\frac{h(x,\vartheta)}{\pm\hbar})$.

Theorem 2.1. *For any $\epsilon > 0$, we have:*

$$(2.5)\qquad \|T_{\lambda,\pm}^{\hbar\,j\,k}\|_{L^2\to L^2} \le c_\epsilon \lambda^{-\frac{n}{2}}\hbar^{-\frac{1}{2}} \qquad \text{for} \quad \hbar > \lambda^{-\frac{1}{2}+\epsilon}2^{\frac{|j|+|k|}{2}},$$

$$(2.6)\qquad \|T_{\lambda,0}^{\hbar\,j\,k}\|_{L^2\to L^2} \le c_\epsilon \lambda^{-\frac{n-1}{2}}\hbar^{\frac{1}{2}}2^{-\frac{|j|+|k|}{2}} \qquad \text{for} \quad \hbar \approx \lambda^{-\frac{1}{2}+\epsilon}2^{\frac{|j|+|k|}{2}}.$$

Corollary 1. *For any* $\epsilon > 0$, $\left\|T_\lambda^{j\,k}\right\|_{L^2\to L^2} \le c_\epsilon \lambda^{-\frac{n}{2}+\frac14+\epsilon} 2^{-\frac{|j|+|k|}{4}}$.

This corollary proves the statement of the Main Theorem.

It is also convenient to have an estimate on the sum of operators with the same values of $\hbar$ and λ, $T_\lambda^\hbar = \sum_{i,j}\left(\sum_\pm T_{\lambda,\pm}^{\hbar,\,j\,k} + T_{\lambda,0}^{\hbar,\,j\,k}\right)$. Although all operators in the right-hand side have similar bounds, (2.5) and (2.6), the number of these operators is bounded by const $\ln\lambda$ (this is due to the inequality $\lambda^{-\frac12+\epsilon}2^{\frac{|j|+|k|}{2}} \le \hbar \le \text{const}$) which can be absorbed into λ^ϵ. Therefore,

Corollary 2. *For any* $\epsilon > 0$, $\left\|T_\lambda^\hbar\right\|_{L^2\to L^2} \le c_\epsilon \lambda^{-\frac{n}{2}+\frac14+\epsilon}$.

We will need this Corollary in Section 4 for $L^p \to L^p$ estimates.

We will prove (2.5) only. The proof of (2.6) is similar but slightly easier, all this according to the scheme in [PhSt 91], [Cu 97], and [Co 99]. It suffices to consider the operator $T_{\lambda,+}^{\hbar\,j\,k}$ with $j,\,k \ge 0$. We can assume that both $2^j\hbar^{1/2}$ and $2^k\hbar^{1/2}$ are smaller than arbitrarily chosen positive constant (if either $|\mathcal{L}h|$ or $|\mathcal{R}h|$ is bounded from below by a positive constant, the corresponding projection from $\mathcal{C}$ is a Whitney fold, and the optimal results follow from [Co 99]).

We decompose the operator $T_\lambda^{\hbar\,j\,k}$ into pieces as follows. Let $\chi \in C_{\text{comp}}^\infty(\mathbb{R}^n)$ be a non-negative smooth function supported near the origin of $\mathbb{R}^n$, such that for any $x \in \mathbb{R}^n$, $\sum_{P\in\mathbb{Z}^n}\chi(x-P) = 1$ (the sum is taken over the points on the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$). For $X, \Theta \in \mathbb{Z}^n$ and for $\sigma > 0$ a small number fixed once for all (we will see later which conditions σ has to satisfy), we define

$$
\begin{aligned}
T_{X\Theta}u(x) = {} & \chi\Big(\frac{x}{2^{-j}\sigma\hbar^{1/2}} - X\Big)\int_{\mathbb{R}^n} e^{i\lambda S(x,\vartheta)}a(x,\vartheta) \\
& \times \beta\Big(\frac{h(x,\vartheta)}{\hbar}\Big)\chi\Big(\frac{\vartheta}{2^{-k}\sigma\hbar^{1/2}} - \Theta\Big)\beta\Big(\frac{\mathcal{R}h(x,\vartheta)}{2^j\hbar^{1/2}}\Big)\beta\Big(\frac{\mathcal{L}h(x,\vartheta)}{2^k\hbar^{1/2}}\Big)u(\vartheta)d\vartheta.
\end{aligned} \tag{2.7}
$$

Inequality (2.5) follows from the Cotlar-Stein almost orthogonality argument applied to the operators $T_{X\Theta}$, which satisfy the following bounds:

Proposition 2.2. *For any* $N > 0$, *there are the following bounds on* L^2-L^2 *norms:*

$$\|T_{X\Theta}\| \le c\lambda^{-\frac{n}{2}}\hbar^{-\frac12}, \tag{2.8}$$

$$\left\|T_{X\Theta}T^*_{\tilde X\tilde\Theta}\right\| + \left\|T^*_{X\Theta}T_{\tilde X\tilde\Theta}\right\| \le \frac{c_{\epsilon,N}\lambda^{-n}\hbar^{-1}}{(1+|X-\tilde X|+|\Theta-\tilde\Theta|)^N}. \tag{2.9}$$

We are going to prove the almost orthogonality relations (2.9), postponing the proof of individual estimates (2.8) until Section 3. It suffices to obtain the bound on $T_{X\Theta}T^*_{\tilde X\tilde\Theta}$. Its Schwartz kernel is given by

$$
\begin{aligned}
K_{X\Theta\tilde X\tilde\Theta}(x,y) = {} & \chi(\frac{x}{2^{-j}\sigma\hbar^{1/2}}-X)\chi(\frac{y}{2^{-j}\sigma\hbar^{1/2}}-\tilde X)\int\limits_{\mathbb{R}^n} e^{i\lambda(S(x,\vartheta)-S(y,\vartheta))} \\
& \times a(x,\vartheta)\bar a(y,\vartheta)\chi(\frac{\vartheta}{2^{-k}\sigma\hbar^{1/2}}-\Theta)\chi(\frac{\vartheta}{2^{-k}\sigma\hbar^{1/2}}-\tilde\Theta)\beta(\frac{h(x,\vartheta)}{\hbar})\beta(\frac{h(y,\vartheta)}{\hbar}) \\
& \times \beta(\frac{\mathcal{R}h(x,\vartheta)}{2^j\hbar^{1/2}})\beta(\frac{\mathcal{R}h(y,\vartheta)}{2^j\hbar^{1/2}})\beta(\frac{\mathcal{L}h(x,\vartheta)}{2^k\hbar^{1/2}})\beta(\frac{\mathcal{L}h(y,\vartheta)}{2^k\hbar^{1/2}})d\vartheta.
\end{aligned}
\tag{2.10}
$$

This is identically zero unless $|\Theta-\tilde\Theta|\le 2\sqrt{n}$; it is not restrictive to assume $\Theta=\tilde\Theta$. We assume that $|X-\tilde X|\ge C_1$, with C_1 large.

The easiest case to consider is $|X'-\tilde X'|\ge|X_n-\tilde X_n|$. According to (2.1) and to (2.3), the matrix $S_{x'\vartheta'}$ is non-degenerate and $|S_{x_n\vartheta'}|\le\varepsilon$, so that

$$
\begin{aligned}
|S_{\vartheta'}(x,\vartheta)-S_{\vartheta'}(y,\vartheta)| &\ge \mathrm{const}\,|x'-y'|-\varepsilon|x_n-y_n| \\
&\approx \sigma\hbar^{\frac12}2^{-j}\left[\mathrm{const}\,|X'-\tilde X'|-\varepsilon|X_n-\tilde X_n|\right].
\end{aligned}
$$

Since $\varepsilon<\frac12$ and $|X'-\tilde X'|\ge|X_n-\tilde X_n|$, we have

$$
|S_{\vartheta'}(x,\vartheta)-S_{\vartheta'}(y,\vartheta)|\ge\mathrm{const}\,\hbar^{\frac12}2^{-j}|X-\tilde X|.
$$

Integrating by parts in the expression for $K_{X\Theta\tilde X\tilde\Theta}(x,y)$, we obtain the bound

$$
|K_{X\Theta\tilde X\tilde\Theta}(x,y)|\le\frac{\mathrm{const}\,(2^{-k}\hbar^{\frac12})^n}{(\lambda\hbar|S_{\vartheta'}(x,\vartheta)-S_{\vartheta'}(y,\vartheta)|)^N}\le\frac{\mathrm{const}\,(2^{-k}\hbar^{\frac12})^n}{(\lambda\hbar^{\frac32}2^{-j}|X-\tilde X|)^N}.
\tag{2.11}
$$

Since $2^j\hbar^{\frac12}\le\mathrm{const}$ and $\hbar\ge\lambda^{-\frac12+\epsilon}$, we have $\lambda\hbar^{\frac32}2^{-j}\ge\mathrm{const}\,\lambda\hbar^2\ge\mathrm{const}\,\lambda^{2\epsilon}$, and then the Schur lemma leads to an estimate which is better than (2.9). (An extra $\hbar$ in the denominator of (2.11) reflects a weak bound $\mathrm{const}\,\hbar^{-1}$ on the contribution from ∂_ϑ during integration by parts. To be able to prove the Main Theorem with $\epsilon=0$, we need a better bound on this contribution.)

Now, instead, we assume that $|X'-\tilde X'|<|X_n-\tilde X_n|$. We shall show that if $|X-\tilde X|\ge C_1$, where C_1 is sufficiently large, then there is some small constant $c_2>0$ (the conditions on both C_1 and c_2 are to be obtained below) such that for all (x,y,ϑ) in the support of the symbol under consideration the following inequality is satisfied:

$$
|S_{\vartheta'}(x,\vartheta)-S_{\vartheta'}(y,\vartheta)|\ge c_2 2^j\hbar^{\frac12}|x_n-y_n|.
\tag{2.12}
$$

If so, the integration by parts in the expression for $K_{X\Theta\tilde X\tilde\Theta}(x,y)$ contributes the factor bounded by

$$
\frac{\mathrm{const}}{(\lambda\hbar\cdot 2^j\hbar^{\frac12}|x_n-y_n|)^N}\approx\frac{\mathrm{const}}{(\lambda\hbar^2|X_n-\tilde X_n|)^N}\le\frac{\mathrm{const}}{(\lambda\hbar^2|X-\tilde X|)^N}.
$$

We apply the Schur lemma to the operator $T_{X\Theta}T^*_{\tilde{X}\tilde{\Theta}}$ and obtain

$$\|T_{X\Theta}T^*_{\tilde{X}\tilde{\Theta}}\| \leq \text{const}\, \frac{(2^{-j}\hbar^{\frac{1}{2}})^n(2^{-k}\hbar^{\frac{1}{2}})^n}{(\lambda\hbar^2|X-\tilde{X}|)^N}. \tag{2.13}$$

Since $\hbar \geq \lambda^{-\frac{1}{2}+\epsilon}2^{\frac{j+k}{2}}$, the denominator is greater than any power of $\lambda^{2\epsilon}|X-\tilde{X}|$, and again we obtain an estimate which is much better than what is needed for (2.9). The requirement of Theorem 2.1 that $\epsilon > 0$ is also crucial here.

Let us now derive conditions on C_1 and c_2. Namely, we will show that if for some small $c_2 > 0$ at some point (x, y, ϑ) on the support of the symbol of (2.10) we have

$$|S_{\vartheta'}(x,\vartheta) - S_{\vartheta'}(y,\vartheta)| < c_2 2^j \hbar^{\frac{1}{2}}|x_n - y_n|, \tag{2.14}$$

then $|X - \tilde{X}|$ has to be uniformly bounded (we will denote this bound by C_1). Therefore, if $|X - \tilde{X}| \geq C_1$, then (2.12) holds for all (x, y, θ) in the support of the integrand, leading to the bound (2.13).

We employ an argument as in Proposition 3.2 in [Co 99]. Assume that $y_n > x_n$ (otherwise we swap x and y). We consider the map

$$\mu:\ \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \qquad \mu:\ (x', x_n, \vartheta) \mapsto (\eta' = S_{\vartheta'}(x,\vartheta), x_n, \vartheta).$$

This map is a diffeomorphism since $\det \frac{\partial(\eta', x_n, \vartheta)}{\partial(x', x_n, \vartheta)} = \det S_{x'\vartheta'}(x,\vartheta) \neq 0$. In the (η', x_n, ϑ)-space, consider the line segment $\mathfrak{l}$ from the point $\mu_0 \equiv \mu(x,\vartheta)$ to $\mu_1 \equiv \mu(y,\vartheta)$, and parameterize it by t so that $t = 0$ at μ_0 and $t = |\mathfrak{l}|$ at μ_1. Let $\mathbf{v}$ be the constant vector field in the (η', x_n, ϑ)-space, generated by the unit vector in the direction of $\mathfrak{l}$. Denoting by $\mathfrak{h}$ the function $h(x,\vartheta)$ rewritten in (η', x_n, ϑ)-variables, so that $h = \mu^*\mathfrak{h}$, we obtain the following inequality:

$$4\hbar \geq |\mathfrak{h}(\mu(y)) - \mathfrak{h}(\mu(x))| = \left|\int_{\mathfrak{l}} \mathbf{v}\mathfrak{h}\, dt\right| \geq |\mu_1 - \mu_0| \cdot \inf_{\mathfrak{l}} |\mathbf{v}\mathfrak{h}|.$$

Since μ is a diffeomorphism, we may assume that $|x - y| \leq \text{const}\,|\mu_1 - \mu_0|$. This leads to the inequality

$$|x - y| \cdot \inf_{\mathfrak{l}} |\mathbf{v}\mathfrak{h}| \leq \text{const}\,\hbar. \tag{2.15}$$

Lemma. *If c_2 in (2.14) is sufficiently small, then $|\mathbf{v}\mathfrak{h}| \geq 2^{j-2}\hbar^{\frac{1}{2}}$ everywhere on $\mathfrak{l}$.*

From the inequality (2.15) and from the above Lemma we conclude that $|x - y|$ is bounded by $\text{const}\, 2^{-j}\hbar^{1/2}$, and this proves that $|X - \tilde{X}|$ is uniformly bounded.

Proof of the Lemma. We denote the partial derivative in the direction x_n in (η', x_n)-space by $\mathbf{r} = \partial_{x_n}\big|_{\eta'}$. Let us notice that $\mathbf{r}$ is a push-forward of the vector field $\mathcal{R}$: for any function f over (η', x_n, ϑ)-space, there is the relation $\mu^*(\mathbf{r}f) = \mathcal{R}(\mu^* f)$, and also $\mathbf{r}\big|_{\mu(\Sigma)} \in \operatorname{Ker} d(\pi_R \circ \mu^{-1})$.

Since the endpoints μ_0 and μ_1 of the arc $\mathfrak{l}$ correspond to the points (x, ϑ) and (y, ϑ) which are on the support of $T_\lambda^{\hbar\, j\, k}$, there are the inequalities

$$\mathbf{r}\mathfrak{h}(\mu_0) = \mathcal{R}h(x, \vartheta) \geq 2^{j-1}\hbar^{\frac{1}{2}}, \qquad \mathbf{r}\mathfrak{h}(\mu_1) = \mathcal{R}h(y, \vartheta) \geq 2^{j-1}\hbar^{\frac{1}{2}}.$$

From (2.14) we conclude that the vector $\mathbf{v}$ is almost parallel to $\mathbf{r}$:

$$|\mathbf{v} - \mathbf{r}| \leq c_2 2^j \hbar^{\frac{1}{2}} \qquad \text{in the Euclidean metric of } (\eta', x_n, \vartheta)\text{-space.} \tag{2.16}$$

Therefore, $|\mathbf{v}\mathfrak{h} - \mathbf{r}\mathfrak{h}| \leq c_2 2^j \hbar^{\frac{1}{2}} |\nabla \mathfrak{h}|$, and if c_2 is sufficiently small, then at the endpoints of $\mathfrak{l}$ one has $\mathbf{v}\mathfrak{h} \geq 2^{j-2}\hbar^{\frac{1}{2}}$. Moreover, the same inequality holds everywhere on $\mathfrak{l}$, since as we will show in a moment the restriction $\mathbf{v}\mathfrak{h}\big|_{\mathfrak{l}}$ is monotone (again, if c_2 is sufficiently small).

It suffices to prove that $\mathbf{v}^2\mathfrak{h} \neq 0$. The map $\pi_R \circ \mu^{-1}$ (like the map π_R) has a singularity of type 2, while on the critical variety of the map $\pi_R \circ \mu^{-1}$ the vector field $\mathbf{r}$ is in the kernel of its differential. Therefore, according to Definition 1.2, we may assume that, on the support of the symbol of T_λ, $\inf |\mathbf{r}^2\mathfrak{h}|$ is uniformly bounded from below by a non-zero constant: $\inf |\mathbf{r}^2\mathfrak{h}| = \kappa > 0$. Applying (2.16), we obtain:

$$|\mathbf{v}^2\mathfrak{h} - \mathbf{r}^2\mathfrak{h}| = |(\mathbf{v} - \mathbf{r})(\mathbf{v} + \mathbf{r})\mathfrak{h}| \leq \operatorname{const} c_2 2^j \hbar^{\frac{1}{2}} \sup |\mathfrak{h}''| \leq \operatorname{const} c_2 \sup |\mathfrak{h}''|.$$

If c_2 is sufficiently small, then the right-hand side is smaller than $\kappa/2$, and then $|\mathbf{v}^2\mathfrak{h}| \geq \kappa/2 > 0$. $\square$

3. Proof of Proposition 2.2: individual estimates

Now we prove the estimate (2.8) on a single piece $T_{X\Theta}$ of the operator $T_\lambda^{\hbar\, j\, k}$,

$$\begin{aligned} T_{X\Theta}u(x) = \chi\Big(\frac{x}{2^{-j}\sigma\hbar^{1/2}} - X\Big) \int\limits_{\mathbb{R}^n} e^{i\lambda S(x,\vartheta)} a(x, \vartheta) \\ \times \beta\Big(\frac{h(x,\vartheta)}{\hbar}\Big) \chi\Big(\frac{\vartheta}{2^{-k}\sigma\hbar^{1/2}} - \Theta\Big) \beta\Big(\frac{\mathcal{R}h(x,\vartheta)}{2^j\hbar^{1/2}}\Big) \beta\Big(\frac{\mathcal{L}h(x,\vartheta)}{2^k\hbar^{1/2}}\Big) u(\vartheta) d\vartheta, \end{aligned} \tag{3.1}$$

for fixed values of $\hbar$, j, k, X, and Θ. Incidentally, in this part of the proof we will not use the condition of Theorem 1.4 that $\epsilon > 0$.

The diameter of the support of the Schwartz kernel of the operator $T_{X\Theta}$ is bounded by $2\sqrt{n}\max\{2^{-j}, 2^{-k}\}\,\sigma\hbar^{1/2}$. Since $dh(x, \vartheta) \neq 0$, the distance from the support of $T_{X\Theta}$ (where $h(x, \vartheta) \approx \hbar$) to the critical variety $\Sigma = \{(x, \vartheta) \mid h(x, \vartheta) = 0\}$

is bounded by const $\hbar$. Therefore, we can pick a point $p_0 = (x_0, \vartheta_0) \in \Sigma$ such that for any (x, ϑ) from the support of the Schwartz kernel of $T_{X\Theta}$

$$|(x,\vartheta) - (x_0,\vartheta_0)| \le 2\sqrt{n}\max\{2^{-j}, 2^{-k}\}\,\sigma\hbar^{1/2} + \text{const}\,\hbar. \tag{3.2}$$

We assume that at the point p_0

$$\partial_{x_n}|_{p_0} \in \operatorname{Ker} d\pi_R, \qquad \partial_{\vartheta_n}|_{p_0} \in \operatorname{Ker} d\pi_L. \tag{3.3}$$

If (3.3) were not satisfied, we would need to change the local coordinates to $(\tilde{x}, \tilde{\vartheta})$, $x = A\tilde{x}$, $\vartheta = B\tilde{\vartheta}$, with A, $B \in SO(n)$, so that (3.3) would be true in the new coordinates: $\partial_{\tilde{x}_n}|_{p_0} \in \operatorname{Ker} d\pi_R$ and $\partial_{\tilde{\vartheta}_n}|_{p_0} \in \operatorname{Ker} d\pi_L$. Then we could use the same argument as will follow below. Before we proceed with the proof, let us verify that all the key inequalities which are satisfied on the support of the Schwartz kernel of $T_{X\Theta}$ would also be satisfied in the new local coordinates.
(i) Since $A \in SO(n)$, the diameter of the $\tilde{x}$-support is the same as the diameter of the x-support and is bounded by $2\sqrt{n}2^{-j}\sigma\hbar^{1/2}$. Similarly, the diameter of the $\tilde{\vartheta}$-support is bounded by $2\sqrt{n}2^{-k}\sigma\hbar^{1/2}$.
(ii) The determinant of the mixed Hessian of S in the old and new coordinates is the same: $\det S_{\tilde{x}\tilde{\vartheta}} = \det A \det B \det S_{x\vartheta} = \det S_{x\vartheta} = h$. Hence, $\frac{\hbar}{2} \le \det S_{\tilde{x}\tilde{\vartheta}} \le 2\hbar$.
(iii) Since $\partial_{\tilde{x}_n}|_{p_0} \in \operatorname{Ker} d\pi_R$, $\partial_{\tilde{\vartheta}_n}|_{p_0} \in \operatorname{Ker} d\pi_L$, and $\operatorname{rank} S_{\tilde{x}\tilde{\vartheta}} = \operatorname{rank} S_{x\vartheta} = n-1$, there is the inequality $\det S_{\tilde{x}'\tilde{\vartheta}'}|_{p_0} \ne 0$. We may assume that $\det S_{\tilde{x}'\tilde{\vartheta}'} \ge \text{const} > 0$.
(iv) Let the vector field $\tilde{\mathcal{L}}$ be given by the same expression as $\mathcal{L}$ but in the new coordinates: $\tilde{\mathcal{L}} = \partial_{\tilde{\vartheta}_n} - S_{\tilde{x}'\tilde{\vartheta}_n} S^{\tilde{\vartheta}'\tilde{x}'}\partial_{\tilde{\vartheta}'}$. We need to check whether the quantity $\tilde{\mathcal{L}}h$ is of the same magnitude as $\mathcal{L}h$ ($\sim 2^k\hbar^{1/2}$). Since both $\mathcal{L}$ and $\tilde{\mathcal{L}}$ restricted onto the critical variety Σ are non-zero and belong to the one-dimensional kernel of $d\pi_L$, there is some smooth non-zero function φ such that the difference $\tilde{\mathcal{L}} - \varphi\mathcal{L}$ vanishes on Σ. Since $dh|_\Sigma \ne 0$, $|\tilde{\mathcal{L}} - \varphi\mathcal{L}| \le O(|h|)$, resulting in $|(\tilde{\mathcal{L}} - \varphi\mathcal{L})h| \le \text{const}\,\hbar$. Therefore, for some constants $C > c > 0$, $c2^k\hbar^{1/2} \le \tilde{\mathcal{L}}h \le C2^k\hbar^{1/2}$. Similarly, there are some constants $C' > c' > 0$ so that $c'2^j\hbar^{1/2} \le \tilde{\mathcal{R}}h \le C'2^j\hbar^{1/2}$, where $\tilde{\mathcal{R}} = \partial_{\tilde{x}_n} - S_{\tilde{x}_n\tilde{\vartheta}'}S^{\tilde{\vartheta}'\tilde{x}'}\partial_{\tilde{x}'}$.

So, we assume that (3.3) is satisfied. Then we conclude that $S_{x_n\vartheta}|_{p_0} = 0$ and $S_{x\vartheta_n}|_{p_0} = 0$. This, in turn, allows to conclude that everywhere on the support of the Schwartz kernel of $T_{X\Theta}$, for some $C > 0$, we have

$$|S_{x\vartheta_n}(x,\vartheta)| + |S_{x_n\vartheta}(x,\vartheta)| \le C\left(\max\{2^{-j}, 2^{-k}\}\,\sigma\hbar^{1/2} + \hbar\right). \tag{3.4}$$

We have applied the bound (3.2) on the distance $|(x,\vartheta) - (x_0,\vartheta_0)|$.

We split $T_{X\Theta}$ into $T_{X\Theta} = \sum_{\alpha' \in \mathbb{Z}^{n-1}} T_{\alpha'}$, with $T_{\alpha'}$ defined by the same expression (3.1) as $T_{X\Theta}$ but with an extra factor $\chi(\frac{x'-\vartheta'}{\sigma\hbar} - \alpha')$. The desired estimate (2.8) on $T_{X\Theta}$ will be a consequence of the following almost orthogonality relations:

Proposition 3.1. *We have the following bounds on the $L^2 \to L^2$ operator norms:*

$$\|T_{\alpha'}\| \leq c\lambda^{-\frac{n}{2}}\hbar^{-\frac{1}{2}}, \tag{3.5}$$

$$\left\|T_{\alpha'}T^*_{\beta'}\right\| + \|T^*_{\alpha'}T_{\beta'}\| \leq c_N\lambda^{-n}\hbar^{-1}(1+|\alpha'-\beta'|)^{-N}. \tag{3.6}$$

We first prove the almost orthogonality relations (3.6). It suffices to bound $T_{\alpha'}T^*_{\beta'}$ (the argument for the other term is the same). Its Schwartz kernel is given by

$$\begin{aligned} K_{\alpha'\beta'}(x,y) &= \chi(\frac{x}{2^{-j}\sigma\hbar^{1/2}} - X)\chi(\frac{y}{2^{-j}\sigma\hbar^{1/2}} - X) \\ &\times \int\limits_{\mathbb{R}^n} e^{i\lambda(S(x,\vartheta)-S(y,\vartheta))}a(x,\vartheta)\bar{a}(y,\vartheta)\beta(\hbar^{-1}h(x,\vartheta)) \\ &\times \beta(\hbar^{-1}h(y,\vartheta))\chi^2(\frac{\vartheta}{2^{-k}\sigma\hbar^{1/2}} - \Theta)\beta(\frac{\mathcal{R}h(x,\vartheta)}{2^j\hbar^{1/2}})\beta(\frac{\mathcal{R}h(y,\vartheta)}{2^j\hbar^{1/2}}) \\ &\times \beta(\frac{\mathcal{L}h(x,\vartheta)}{2^k\hbar^{1/2}})\beta(\frac{\mathcal{L}h(y,\vartheta)}{2^k\hbar^{1/2}})\chi(\frac{x'-\vartheta'}{\sigma\hbar} - \alpha')\chi(\frac{y'-\vartheta'}{\sigma\hbar} - \beta')d\vartheta. \end{aligned} \tag{3.7}$$

Let us consider the Taylor expansion of $S_{\vartheta'}(x,\vartheta) - S_{\vartheta'}(y,\vartheta)$,

$$(x'-y')\cdot S_{x'\vartheta'}(x,\vartheta) + (x_n - y_n)S_{x_n\vartheta'}(x,\vartheta) + O(|x-y|^2). \tag{3.8}$$

Since $\det S_{x'\vartheta'} \neq 0$, the first term in (3.8) is not smaller than const $|x'-y'|$. Since $|x'-y'| > (|\alpha'-\beta'| - 2\sqrt{n-1})\sigma\hbar$, $|x-y| \leq \text{const}\,\sigma\hbar^{1/2}$, and due to the inequality (3.4), we can pick σ small enough and require that $|\alpha'-\beta'|$ be large enough so that the second and the third terms in (3.8) are dominated by the first term. Therefore,

$$|S_{\vartheta'}(x,\vartheta) - S_{\vartheta'}(y,\vartheta)| \geq \text{const}\,|x'-y'|.$$

Using the above inequality, we integrate by parts in (3.7) and obtain the following bound on the Schwartz kernel of $T_{\alpha'}T^*_{\beta'}$:

$$|K_{\alpha'\beta'}(x,y)| \leq \text{const} \int\limits_{\mathbb{R}^n} \left[1 + \lambda\hbar|x'-y'|\right]^{-N} (\lambda\hbar^2)^{-N}|\alpha'-\beta'|^{-N}d\vartheta.$$

We apply the Schur lemma. The integration in ϑ contributes $\hbar^{n-1}\cdot 2^{-k}\hbar^{1/2}$. The integration in x (or y) contributes $(\lambda\hbar)^{-(n-1)}\cdot 2^{-j}\hbar^{1/2}$. This results in

$$\int\limits_{\mathbb{R}^n} |K_{\alpha'\beta'}(x,y)|dx \leq \text{const}\,\lambda^{-(n-1)}2^{-j-k}\hbar\cdot(\lambda\hbar^2)^{-1}|\alpha'-\beta'|^{-N},$$

proving (3.6).

Now we focus on the estimate (3.5) on a single $T_{\alpha'}$. We assume that $j \geq k$, so that $\max\{2^{-j}, 2^{-k}\} = 2^{-k}$, and therefore (3.4) becomes

$$\sup |S_{x_n \vartheta}| + \sup |S_{x \vartheta_n}| \leq C 2^{-k} \hbar^{1/2} (\sigma + \hbar^{1/2}). \tag{3.9}$$

We consider the kernel of $T_{\alpha'} T^*_{\alpha'}$ (if $j < k$, we consider $T^*_{\alpha'} T_{\alpha'}$ instead):

$$K_{\alpha'\alpha'}(x, y) = \chi(\frac{x}{2^{-j}\sigma\hbar^{1/2}} - X)\chi(\frac{y}{2^{-j}\sigma\hbar^{1/2}} - X) \int\limits_{\mathbb{R}^n} e^{i\lambda(S(x,\vartheta)-S(y,\vartheta))} \mathcal{A} d\vartheta \tag{3.10}$$

where $\mathcal{A}(x, y, \vartheta)$ is given by

$$\begin{aligned} \mathcal{A} = a(x, \vartheta)\bar{a}(y, \vartheta)\beta(\hbar^{-1}h(x, \vartheta))\beta(\hbar^{-1}h(y, \vartheta))\chi^2(\frac{\vartheta}{2^{-k}\sigma\hbar^{1/2}} - \Theta)\beta(\frac{\mathcal{R}h(x, \vartheta)}{2^j\hbar^{1/2}}) \\ \times\beta(\frac{\mathcal{R}h(y, \vartheta)}{2^j\hbar^{1/2}})\beta(\frac{\mathcal{L}h(x, \vartheta)}{2^k\hbar^{1/2}})\beta(\frac{\mathcal{L}h(y, \vartheta)}{2^k\hbar^{1/2}})\chi(\frac{x' - \vartheta'}{\sigma\hbar} - \alpha')\chi(\frac{y' - \vartheta'}{\sigma\hbar} - \alpha'). \end{aligned}$$

On the support of $\mathcal{A}$ we have

$$|x' - y'| \leq 2\sigma\hbar\sqrt{n-1}, \qquad |x_n - y_n| \leq 2^{1-j}\sigma\hbar^{1/2}. \tag{3.11}$$

To integrate by parts in (3.10), we introduce the operators

$$\begin{aligned} \mathcal{M} &= \frac{S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)}{i\lambda|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)|^2} \cdot \nabla_\vartheta, \\ \mathcal{M}_n &= \frac{1}{i\lambda(S_{\vartheta_n}(x, \vartheta) - S_{\vartheta_n}(y, \vartheta))} \frac{\partial}{\partial\vartheta_n}. \end{aligned}$$

Lemma 1. *If*

$$|x_n - y_n| \leq \alpha 2^k \hbar^{-1/2} |x' - y'|, \tag{3.12}$$

with $\alpha > 0$ sufficiently small, then there is a bound

$$|(\mathcal{M}^*)^N \mathcal{A}| \leq C_{N,\alpha}(\lambda\hbar|S_\vartheta(x, \vartheta) - S_\vartheta(y, \vartheta)|)^{-N} \leq C_{N,\alpha}(\lambda\hbar|x' - y'|)^{-N}. \tag{3.13}$$

Lemma 2. *If*

$$|x_n - y_n| \geq \alpha 2^k \hbar^{-1/2} |x' - y'|, \tag{3.14}$$

for the same constant $\alpha > 0$ as in Lemma 1, then

$$|(\mathcal{M}_n^*)^N \mathcal{A}| \leq C_{N,\alpha}(\lambda\hbar^{\frac{3}{2}} 2^{-k}|x_n - y_n|)^{-N}. \tag{3.15}$$

Let us show how to obtain an estimate on $T_{\alpha'}T^*_{\alpha'}$ from two above lemmas. If $|x_n - y_n| < \alpha 2^k \hbar^{-1/2}|x' - y'|$, we integrate in (3.10) by parts with the aid of $\mathcal{M}$, applying Lemma 1; in the opposite case, we integrate by parts with the aid of $\mathcal{M}_n$ and apply Lemma 2. In both cases, we obtain the same bound on the Schwartz kernel of $T_{\alpha'}T^*_{\alpha'}$:

$$|K_{\alpha'\alpha'}(x,y)| \le \int\limits_{\mathbb{R}^n} \chi(\frac{x}{2^{-j}\sigma\hbar^{1/2}} - X)\chi(\frac{y}{2^{-j}\sigma\hbar^{1/2}} - X)\chi^2(\frac{\vartheta}{2^{-k}\sigma\hbar^{1/2}} - \Theta)$$
$$\times \chi(\frac{x' - \vartheta'}{\sigma\hbar} - \alpha')\chi(\frac{y' - \vartheta'}{\sigma\hbar} - \alpha') \left[1 + \lambda\hbar\left(|x' - y'| + \hbar^{\frac{1}{2}} 2^{-k}|x_n - y_n|\right)\right]^{-N} d\vartheta.$$

We apply the Schur lemma. The integration in x' (or in y') contributes $(\lambda\hbar)^{-(n-1)}$, the integration in x_n (or in y_n) contributes $(\lambda\hbar^{\frac{3}{2}}2^{-k})^{-1}$, the integration in ϑ' contributes $\hbar^{n-1}$, and the integration in ϑ_n contributes $2^{-k}\hbar^{1/2}$. We conclude that the $L^2 \to L^2$ norm of $T_{\alpha'}T^*_{\alpha'}$ is bounded by $\mathrm{const}\,\lambda^{-n}\hbar^{-1}$, which proves (3.5). This completes the proof of Proposition 3.1.

Proof of Lemma 1. First, we claim that if (3.12) is true, then

$$|S_\vartheta(x,\vartheta) - S_\vartheta(y,\vartheta)| \ge \mathrm{const}\,|x' - y'|. \tag{3.16}$$

For this, we expand $S_{\vartheta'}(x,\vartheta) - S_{\vartheta'}(y,\vartheta)$ into the Taylor series:

$$(x' - y')\cdot S_{x'\vartheta'}(x,\vartheta) + (x_n - y_n)S_{x_n\vartheta'}(x,\vartheta) + O(|x - y|^2). \tag{3.17}$$

The magnitude of the first term in (3.17) is not smaller than const $|x'-y'|$, while the latter quantity dominates two other terms if α and σ are small (we can bound both $|x_n-y_n|\cdot|S_{x_n\vartheta'}|$ and $|x-y|\cdot|x-y|$ by $\mathrm{const}\,(\alpha 2^k\hbar^{-1/2}+1)|x'-y'|\cdot 2^{-k}\hbar^{1/2}(\sigma+\hbar^{1/2})$).

Now we need to check that each time during integration by parts in (3.10) with the aid of the operator $\mathcal{M}$, the contribution of the derivative ∂_ϑ is bounded by $\mathrm{const}\,\hbar^{-1}$. This includes the case when ∂_ϑ acts on the denominator of $\mathcal{M}$, since due to (3.12) and (3.16),

$$\left|\nabla_\vartheta \frac{1}{|S_\vartheta(x,\vartheta) - S_\vartheta(y,\vartheta)|}\right| \le \frac{\mathrm{const}\,|x-y|}{|S_\vartheta(x,\vartheta) - S_\vartheta(y,\vartheta)|^2} \le \frac{\mathrm{const}}{\hbar|S_\vartheta(x,\vartheta) - S_\vartheta(y,\vartheta)|}.$$

This proves Lemma 1. □

Proof of Lemma 2. We will need the following bound for the denominator of $\mathcal{M}_n$:

$$|S_{\vartheta_n}(x,\vartheta) - S_{\vartheta_n}(y,\vartheta)| \ge \mathrm{const}\,\hbar|x_n - y_n|. \tag{3.18}$$

To prove (3.18), we use the Taylor expansion for $S_{\vartheta_n}(x,\vartheta) - S_{\vartheta_n}(y,\vartheta)$:

$$(3.19)\quad \begin{aligned} &(x_n - y_n)S_{x_n\vartheta_n}(x,\vartheta) + (x'-y')S_{x'\vartheta_n}(x,\vartheta)\\ &+ \frac{(x_n-y_n)^2 S_{x_nx_n\vartheta_n}(x,\vartheta)}{2} + O(|x'-y'|^2 + |x'-y'||x_n-y_n| + |x-y|^3). \end{aligned}$$

Due to (3.9), $\det S_{x\vartheta} = S_{x_n\vartheta_n}\det S_{x'\vartheta'} + O(2^{-2k}\hbar(\sigma+\hbar^{1/2})^2)$, and we conclude that $|S_{x_n\vartheta_n}| \geq \mathrm{const}\,|\det S_{x\vartheta}| \geq \mathrm{const}\,\hbar$ (provided that σ is sufficiently small and also that $\hbar \leq \sigma^2$). From (3.9) and (3.14), we see that the second term in (3.19) is bounded by $\mathrm{const}\,\hbar|x_n - y_n|(\sigma + \hbar^{1/2})$, which is dominated by the first term in (3.19). To deal with the third term in (3.19), we need to bound $|S_{x_nx_n\vartheta_n}(x,\vartheta)|$. Differentiating $h = \det S_{x\vartheta}$ by x_n, we obtain

$$(3.20)\quad \partial_{x_n}h = \det\begin{bmatrix} S_{x'\vartheta'} & S_{x_nx_n\vartheta'}\\ S_{x'\vartheta_n} & S_{x_nx_n\vartheta_n}\end{bmatrix} + O(\hbar^{1/2}) = S_{x_nx_n\vartheta_n}\det S_{x'\vartheta'} + O(\hbar^{1/2}),$$

where we used the bounds (3.9). On the other hand, due to the bound $|\mathcal{R}h| \leq 2^{j+1}\hbar^{1/2}$ at the points (x,ϑ) and (y,ϑ), we derive that at both these points

$$(3.21)\quad |\partial_{x_n}h| = |\mathcal{R}h + S_{x_n\vartheta'}S^{\vartheta'x'}\partial_{x'}h| \leq |\mathcal{R}h| + O(\hbar^{\frac{1}{2}}) \leq \mathrm{const}\,2^j\hbar^{1/2},$$

where we also used the inequality (3.9). Comparing (3.20) and (3.21), we see that

$$(3.22)\quad |S_{x_nx_n\vartheta_n}| \leq \mathrm{const}\,2^j\hbar^{1/2}.$$

Recalling (3.11), we conclude that the third term in (3.19) is also dominated by $\mathrm{const}\,\hbar|x_n-y_n|$ (if σ is sufficiently small). The inequalities (3.11) and (3.14) show that the last term in (3.19) is also dominated by $\mathrm{const}\,\hbar|x_n-y_n|$ (if σ is small).

To complete the proof of Lemma 2, we need to show that each time when we integrate by parts with the aid of $\mathcal{M}_n$, the derivative ∂_{ϑ_n} contributes factors bounded by $\mathrm{const}\,2^k\hbar^{-1/2}$. Indeed, when ∂_{ϑ_n} acts on the cut-off functions $\beta(\frac{\mathcal{L}h}{2^k\hbar^{1/2}})$ and $\beta(\frac{\mathcal{R}h}{2^j\hbar^{1/2}})$, one gets a contribution bounded by $\mathrm{const}\,\hbar^{-1/2}$. When ∂_{ϑ_n} acts on $\beta(\hbar^{-1}h)$, we are getting $\hbar^{-1}\partial_{\vartheta_n}h$, while $|\partial_{\vartheta_n}h| = |\mathcal{L}h + S_{x'\vartheta_n}S^{\vartheta'x'}\partial_{\vartheta'}h|$ is bounded by $|\mathcal{L}h| + O(\hbar^{\frac{1}{2}}) \leq \mathrm{const}\,2^k\hbar^{1/2}$ at both (x,ϑ) and (y,ϑ). This is similar to (3.21). When ∂_{ϑ_n} acts on the denominator of $\mathcal{M}_n$, we get

$$(3.23)\quad \left|\frac{\partial}{\partial\vartheta_n}\left(\frac{1}{|S_{\vartheta_n}(x,\vartheta) - S_{\vartheta_n}(y,\vartheta)|}\right)\right| = \frac{|S_{\vartheta_n\vartheta_n}(x,\vartheta) - S_{\vartheta_n\vartheta_n}(y,\vartheta)|}{|S_{\vartheta_n}(x,\vartheta) - S_{\vartheta_n}(y,\vartheta)|^2}.$$

Due to (3.18), the denominator is not smaller than $\hbar|x-y|$. To deal with the Taylor expansion of the numerator,

$$(x_n-y_n)S_{x_n\vartheta_n\vartheta_n}(x,\vartheta) + (x'-y')\cdot S_{x'\vartheta_n\vartheta_n}(x,\vartheta) + O(|x-y|^2),$$

we bound $|S_{x_n\vartheta_n\vartheta_n}(x,\vartheta)|$ by $\mathrm{const}\,2^k\hbar^{1/2}$ (similarly to (3.22)). This, together with the inequalities (3.11) and (3.14), allows to bound (3.23) by $\mathrm{const}\,2^k\hbar^{-1/2}$, finishing the proof of Lemma 2. $\square$

4. L^p ESTIMATES

Let us give a sketch of the derivation of L^p estimates stated in Theorem 1.1. Given a singular Fourier integral operator $\mathfrak{F} \in I^\mu(X,Y,\mathcal{C})$, we construct the operators $\mathfrak{F}_+^\hbar$ and $\mathfrak{F}_-^\hbar$ by localizing the symbol of $\mathfrak{F}$ (with the aid of dyadic partition of unity) to the variety where the determinant of the Jacobi matrix of $\pi_{_L} : \mathcal{C} \to T^*X$ (computed in some fixed coordinate system) takes values between $\hbar/2$ and $2\hbar$ and between $-2\hbar$ and $-\hbar/2$, respectively, with $0 < \hbar < 1$. The value of the determinant of the Jacobi matrix of $\pi_{_R}$ is then also of magnitude $\hbar$. We decompose $\mathfrak{F}$ into

$$\mathfrak{F} = \mathfrak{F}_0 + \sum_{\pm} \sum_{\hbar = 2^{-N},\, N\in\mathbb{N}} \mathfrak{F}_\pm^\hbar, \tag{4.1}$$

where $\mathfrak{F}_0$ is a classical Fourier integral operator (associated to a local graph).

Assume that $\mathfrak{F} \in I^{-\epsilon}(X,Y,\mathcal{C})$, for any $\epsilon > 0$. We also assume that the symbol of $\mathfrak{F}$ is compactly supported in $X \times Y$. If both projections $\pi_{_L} : \mathcal{C} \to T^*X$ and $\pi_{_R} : \mathcal{C} \to T^*Y$ are cusps, then Corollary 2 after Theorem 2.1 leads us via the standard arguments to the conclusion that

$$\|\mathfrak{F}_\pm^\hbar\|_{L^2(Y)\to L^2(X)} \le \mathrm{const}\, \hbar^{-\frac{1}{2}}. \tag{4.2}$$

Now, assume that $\mathfrak{F} \in I^{-\frac{n-1}{2}}(X,Y,\mathcal{C})$, and that its symbol is compactly supported in $X \times Y$. If the projections $\mathcal{C} \to T^*X$ and $\mathcal{C} \to T^*Y$ are cusps and if the projections $\mathcal{C} \to X$ and $\mathcal{C} \to Y$ are submersions, then for any $\epsilon > 0$

$$\|\mathfrak{F}_\pm^\hbar\|_{\mathrm{H}^1(Y)\to L^1(X)} \le \mathrm{const}_{\,\epsilon} \hbar^{\frac{1}{w_R}-\epsilon}. \tag{4.3}$$

Here $w_{_R}$ is equal to 1 if $\pi_{_R}$ is a strong cusp and $w_{_R} = 2$ otherwise. The proof of (4.3) immediately follows from [CoCu 98].

Assume that $\mathfrak{F} \in I^{-\alpha_p-\epsilon}(X,Y,\mathcal{C})$, for any $\epsilon > 0$, where $\alpha_p = (n-1)|\frac{1}{p} - \frac{1}{2}|$, and that the symbol of $\mathfrak{F}$ is compactly supported in $X \times Y$. If, as above, the projections $\mathcal{C} \to T^*X$ and $\mathcal{C} \to T^*Y$ are cusps and the projections $\mathcal{C} \to X$ and $\mathcal{C} \to Y$ are submersions, then by interpolation of (4.2) and (4.3) we derive that $\mathfrak{F}_\pm^\hbar$ is continuous from $L^p(Y)$ to $L^p(X)$, with its norm proportional to a positive power of $\hbar$ if $1 < p < \frac{w+2}{w+1}$. Thus, $\|\mathfrak{F}\|_{L^p\to L^p} \le \sum_\pm \sum_\hbar \|\mathfrak{F}_\pm^\hbar\|_{L^p\to L^p}$ is bounded by a convergent geometric series, and therefore $\mathfrak{F}$ is continuous from $L^p(Y)$ to $L^p(X)$.

For $\frac{w+2}{w+1} \le p < 2$, the estimates are obtained by interpolation with the L^2 estimates stated in Theorem 1.1. For $p > 2$ the estimates are obtained by duality.

ACKNOWLEDGMENTS

The authors are grateful to their advisor, Professor D.H. Phong, for introduction to the field of oscillatory integral operators and for the encouragement.

References

[Co 99] A. Comech, *Optimal estimates for Fourier integral operators with one-sided folds*, Comm. Partial Differential Equations **24** (1999), 1263–1281.

[CoCu 98] A. Comech, S. Cuccagna, *On L^p continuity of singular Fourier integral operators*, preprint, http://www.math.sunysb.edu/˜comech/papers/hardy.dvi (1998).

[Cu 97] S. Cuccagna, *L^2 estimates for averaging operators along curves with two sided k fold singularities*, Duke Journal **89** (1997), 203–216.

[GrSe 94] A. Greenleaf and A. Seeger, *Fourier integral operators with fold singularities*, J. Reine Angew. Math. **455** (1994), 35–56.

[GrSe 98] ———, *Fourier integral operators with cusp singularities*, Amer. J. Math. **120** (1998), 1077–1119.

[GrSeW 97] A. Greenleaf, A. Seeger, S. Wainger, *Estimates for generalized Radon transforms in three and four dimensions*, Contemporary Mathematics, Proceedings of the Ehrenpreis conference (to appear).

[GrU 91] A. Greenleaf and G. Uhlmann, *Composition of some singular Fourier integral operators and estimates for restricted X-ray transforms. II*, Duke Math. J. **64** (1991), 415–444.

[Hö 71] L. Hörmander, *Fourier integral operators*, Acta Math. **127** (1971), 79–183.

[MT 85] R.B. Melrose and M.E. Taylor, *Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle*, Adv. in Math. **55** (1985), 242–315.

[Mo 65] B. Morin, *Canonical forms of the singularities of a differentiable mapping*, C. R. Acad. Sci. Paris **260** (1965), 6503–6506.

[Ph 94] D.H. Phong, *Singular integrals and Fourier integral operators*, Essays on Fourier Analysis in honor of Elias M. Stein (C. Fefferman, R. Fefferman, and S. Wainger, eds.), Princeton Univ. Press, 1994, pp. 287–320.

[PhSt 91] D.H. Phong and E.M. Stein, *Radon transform and torsion*, Internat. Math. Res. Notices **4** (1991), 49–60.

[PhSt 97] ———, *Newton polyhedron and oscillatory integral operators*, Acta Mat. **179** (1997), 105–152.

[Ry 99] V. Rychkov, *Sharp L^2 bounds for oscillatory integral operators with C^∞ phases*, Math. Z (to appear).

[Se 98] A. Seeger, *Radon transforms and finite type conditions*, J. Amer. Math. Soc. **4** (1998), 869–897.

[SeSoSt 91] A. Seeger, C.D. Sogge and E.M. Stein, *Regularity properties of Fourier integral operators*, Ann. of Math. **134** (1991), 231–251.

[SmSo 94] H.F. Smith and C.D. Sogge, *L^p regularity for the wave equation with strictly convex obstacles*, Duke Math. J. **73** (1994), 97–153.

[T 98] D. Tataru, *On the regularity of boundary traces for the wave equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), 185–206.

State University of New York at Stony Brook, Stony Brook, NY 11794

New York University, New York, NY 10012