Regularizations and Rank One Perturbations

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Outline

Rank one perturbations and spectral representation V
- Introduction
- Representation formula for V
- Rigidity theorem

Singular integral operators
- Singular integral operator V
- Uniform bounds for regularizations

Absence of singular spectrum

Setting

- A self-adjoint operator on Hilbert space $\mathcal{H}$, some vector φ
- Here for simplicity A bounded and $\varphi \in \mathcal{H}$
- Consider family of rank one perturbations

$$A + \alpha(\,\cdot\,, \varphi)\varphi \text{ for } \alpha \in \mathbb{R}$$

- WLOG $\varphi \in \mathcal{H}$ cyclic for A, i.e. $\mathcal{H} = \overline{\operatorname{span}\{A^n\varphi : n \in \mathbb{N}\}}$
- Question: How stable is the embedded singular spectrum when we change α?

Reformulation using the spectral theorem

- $\mu := \mu^\varphi$ denotes the unique spectral measure of A wrt φ, i.e. $(A^n\varphi, \varphi) = \int t^n d\mu(t)$ for $n \in \mathbb{N}$
- There exists a unitary operator $U : \mathcal{H} \to L^2(\mu)$ such that $UA = M_t U$ and $U\varphi = \mathbf{1}$. Notation:

$$A, \varphi \text{ on } \mathcal{H} \quad \overset{U}{\sim} \quad M_t, \mathbf{1} \text{ on } L^2(\mu)$$

- $A + \alpha(\,\cdot\,,\varphi)\varphi$ on $\mathcal{H}$ $\overset{U}{\sim}$ $A_\alpha := M_t + \alpha(\,\cdot\,,\mathbf{1})\mathbf{1}$ on $L^2(\mu)$

Definition of V

- Recall $A_\alpha = M_t + \alpha(\,\cdot\,, \mathbf{1}_t)\mathbf{1}_t$ on $L^2(\mu)$
- For the perturbed operator

$$A_\alpha, \mathbf{1}_t \text{ on } L^2(\mu) \quad \overset{V=V_\alpha}{\sim} \quad M_s, \mathbf{1}_s \text{ on } L^2(\mu_\alpha),$$

  i.e. for some unitary operator $V = V_\alpha : L^2(\mu) \to L^2(\mu_\alpha)$ we have $M_s V = V A_\alpha$ and $V\mathbf{1}_t = \mathbf{1}_s$
- μ_α contains all the spectral information of A_α, because vector $\mathbf{1}_t$ can be shown to be cyclic for A_α

Statement

Theorem (Representation formula)

Under the all the above assumptions, the unitary operator $V : L^2(\mu) \to L^2(\mu_\alpha)$, such that $M_s V = V A_\alpha$ is given by

$$V f(s) = f(s) - \alpha \int \frac{f(s) - f(t)}{s - t} \, d\mu(t)$$

for all compactly supported C^1 functions f.

Statement

Theorem (Rigidity theorem)

Suppose μ *on* $\mathbb{R}$ *is not a single atom and satisfies* $\int (1+|t|)^{-1}\,d\mu(t) < \infty$. *Assume that*

$$Tf(s) = f(s) - \alpha \int \frac{f(s)-f(t)}{s-t}\,d\mu(t) \quad \text{for} \quad f \in C_0^1$$

extends to a bounded operator $L^2(\mu) \to L^2(\nu)$ *with* $\operatorname{Ker} T = \{0\}$.

Then there exists a function h *such that* $1/h \in L^\infty(\nu)$ *and such that* $M_h T : L^2(\mu) \to L^2(\nu)$ *is unitary.*

Moreover the unitary operator $U = M_h T$ *gives the spectral representation of the operator* $A_\alpha := M_t + \alpha(\,\cdot\,, \mathbf{1}_t)\mathbf{1}_t$ *in* $L^2(\mu)$, *namely* $UA_\alpha = M_s U$.

Idea of proof

- Recall from linear algebra: If $AT = TB$ for hermitian$A$, B and $\operatorname{Ker} T = \operatorname{Ker} T^* = \{0\}$, then $A \sim B$
- Can prove $M_s T = T[M_t + \alpha(\,\cdot\,, \mathbf{1}_t)\mathbf{1}_t]$ and $\operatorname{Ker} T^* = \{0\}$
- Assumed $\operatorname{Ker} T = \{0\}$
- Conclusion $M_h T$ unitary for some h with $1/h \in L^\infty(\nu)$

Singular integral operator V

- f and g have separated compact supports, i.e. both $\operatorname{supp} f$ and $\operatorname{supp} g$ are compact and $\operatorname{dist}(\operatorname{supp} f, \operatorname{supp} g) > 0$
- Bounded operator $T : L^2(\mu) \to L^2(\nu)$ is a SIO, if

$$(Tf, g)_{L^2(\nu)} = \int\int K(s,t) f(t) \overline{g(s)}\, d\mu(t)\, d\nu(s)$$

for all $f \in L^2(\mu)$, $g \in L^2(\nu)$ with separated compact supports

Lemma

Operator $V : L^2(\mu) \to L^2(\mu_\alpha)$ from the representation theorem is a SIO with kernel $K(s,t) = -\alpha(s-t)^{-1}$.

In particular, $T_\mu := \alpha^{-1} V : L^2(\mu) \to L^2(\mu_\alpha)$ is a SIO with kernel $(t-s)^{-1}$ and $\|T_\mu\|_{L^2(\mu) \to L^2(\mu_\alpha)} \le |\alpha|^{-1}$.

Introduction

- For $\varepsilon > 0$ consider regularized operators

$$T_\varepsilon f(s) = \int \frac{f(t)}{s - t + i\varepsilon}\, d\mu(t) \quad \text{and}$$
$$\widetilde{T}_\varepsilon f(s) = \int_{|t-s|>\varepsilon} \frac{f(t)}{s-t}\, d\mu(t)$$

- T_ε, $\widetilde{T}_\varepsilon$ well defined for compactly supported f
- If $\int (1+x^2)^{-1}\, d\mu(x) < \infty$, then the operators are well defined by the above formulas for all $f \in L^2(\mu)$

Statement

Theorem (Regularizations)

Let μ and ν be Radon measures such that their singular parts are mutually singular, i.e. $\mu_{\mathrm{s}} \perp \nu_{\mathrm{s}}$, and such that

$$\left| \iint \frac{f(t)\overline{g(s)}}{s-t}\, d\mu(t)\, d\nu(s) \right| \leq C \|f\|_{L^2(\mu)} \|g\|_{L^2(\nu)}$$

for all $f \in L^2(\mu)$ and $g \in L^2(\nu)$ with separated compact supports.

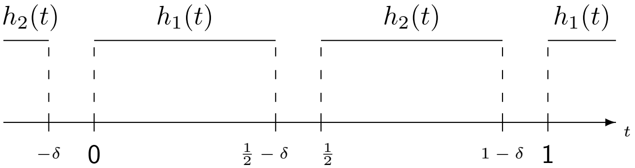
Then operators $T_\varepsilon, \widetilde{T}_\varepsilon : L^2(\mu) \to L^2(\nu)$ are uniformly bounded.

- Classical theory of SIO does not apply, because μ is allowed to be non-doubling.

Proof: Regularizations (for T_ε and a.c. μ, ν)

- For f, g with separated compact supports and uniformly in ε

$$\left| \iint \frac{f(t)\overline{g(s)}}{s-t+i\varepsilon}\, d\mu(t) d\nu(s) \right| \le 2C \|f\|_\mu \|g\|_\nu$$



- Here δ small
- Extend h_1, h_2 to 1-periodic functions on $\mathbb{R}$
- Take $f \in L^2(\mu)$ and $g \in L^2(\nu)$
- Take $f_n(t) := f(t)h_1(nt)$ and $g_n(s) := g(s)h_2(ns)$
- For each n functions f_n and g_n of separated compact support

Proof: Regularizations (for T_ε and a.c. μ, ν)

Claims

- $f_n \to (1/2-\delta)f$ and $g_n \to (1/2-\delta)g$ weakly in $L^2(\mu)$ and $L^2(\nu)$ respectively, and
- $\|f_n\|_\mu^2 \to (1/2-\delta)\|f\|_\mu^2$ and $\|g_n\|_\nu^2 \to (1/2-\delta)\|g\|_\nu^2$

hold true, because for $h = h_1$ or $h = h_2$

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi(t)h(nt)dt = (1/2-\delta)\int_{\mathbb{R}} \phi(t)dt \text{ for all } \phi \in L^1.$$

Since T_ε is compact, we have

$$\begin{aligned}(1/2-\delta)^2|(T_\varepsilon f, g)| &= \lim_{n\to\infty} |(T_\varepsilon f_n, g_n)| \\ &\le C \lim_{n\to\infty} \|f_n\|_\mu \|g_n\|_\nu = (1/2-\delta)C\|f\|_\mu\|g\|_\nu,\end{aligned}$$

so $\|T_\varepsilon\| \le (1/2-\delta)^{-1}C$.

Proof: Regularizations (for T_ε and general μ, ν)

- Disjoint E, F so that $|E| = |F| = 0$, $\mu_{\text{s}}(E^c) = 0$, $\nu_{\text{s}}(F^c) = 0$
- $G := (E \cup F)^c$
- Take compact $E_n \subset E$, $F_n \subset F$, $G_n \subset G$ such that

$$\mu(E_n) \to \mu(E), \nu(F_n) \to \nu(F), \mu(G_n) + \nu(G_n) \to \mu(G) + \nu(G)$$

- $f_a := f\chi_G$ and $g_a := g\chi_G$ and $f_s := f\chi_E$ and $g_s := g\chi_F$
- For $\delta > 0$ let

$$f_n(t) := f_a(t)h_1(nt)\chi_{G_n}(t) + (1/2 - \delta)f_s(t)\chi_{E_n}(t)$$
$$g_n(t) := g_a(t)h_2(nt)\chi_{G_n}(t) + (1/2 - \delta)g_s(t)\chi_{F_n}(t)$$

This completes the proof of the Regularization Theorem for T_ε.

Proof: Regularizations (for truncated operators $\widetilde{T}_\varepsilon$)

- Consider $(T_\varepsilon - \widetilde{T}_\varepsilon)f(s) = \int K_\varepsilon(s-t)f(t)d\mu(t)$ with $K_\varepsilon(x) = (x+i\varepsilon)^{-1} - x^{-1}\chi_{[-\varepsilon,\varepsilon]^c}$
- For each ε choose intervals I_k and $c_k > 0$ such that $\sum c_k \le C < \infty$ and $|K_\varepsilon(t)| \le \sum c_k |I_k|^{-1}\chi_{I_k}(t)$
- Remains uniform boundedness of averaging operator

$$T_I f(s) = |I|^{-1} \int \chi_I(s-t)f(t)d\mu(t)$$

- Uniform boundedness of T_ε implies Muckenhoupt condition

$$|I|^{-2}\mu(I)\nu(I) \le C$$

□

Statement

Theorem (Regularizations)

Let μ and ν be Radon measures such that their singular parts are mutually singular, i.e. $\mu_{\rm s} \perp \nu_{\rm s}$, and such that

$$\left| \iint \frac{f(t)\overline{g(s)}}{s-t}\, d\mu(t)\, d\nu(s) \right| \leq C \|f\|_{L^2(\mu)} \|g\|_{L^2(\nu)}$$

for all $f \in L^2(\mu)$ and $g \in L^2(\nu)$ with separated compact supports.

Then operators $T_\varepsilon, \widetilde{T}_\varepsilon : L^2(\mu) \to L^2(\nu)$ are uniformly bounded.

Alternative representation formula

- Assume the setting of rank one perturbations
- $(T_\mu)_\varepsilon : L^2(\mu) \to L^2(\mu_\alpha)$ is SIO with kernel $(s - t + i\varepsilon)^{-1}$

Theorem (Alternative representation formula)

The weak limit T of T_ε exists as $\varepsilon \to 0^+$, and V satisfies

$$Vf(s) = f(s)(\mathbf{1} - \alpha T\mathbf{1}) + \alpha Tf \quad \textit{for} \quad f \in L^2(\mu).$$

- Notice that $T\mathbf{1} = \frac{1}{\alpha}$ on $(\mu_\alpha)_{\mathrm{s}}$
- Formally in accordance with normalized Cauchy transform from Clark theory

Absence of singular spectrum

- $A_\alpha = M_t + \alpha(\,\cdot\,, \mathbf{1}_t)\mathbf{1}_t$, consider Lebesgue decomposition $d\mu = w dt + d\mu_{\mathrm{s}}$ of A's spectral measure
- Distribution function $D_w(t) := |\{w < t\} \cap I|$ and its inverse function, the increasing rearrangement w^* of w on I, i.e. $w^* := (D_w)^{-1}$

Theorem (Absence of singular spectrum)

If for a bounded open interval I

$$\int_0^\varepsilon x^{-2} w_I^*(x)\, dx = \infty \text{ or } \int_0^\varepsilon 1/D_w(t)\, dt = \infty \text{ or } 1/w \in L^{1,\infty}_{\mathrm{loc}}(I)$$

for some (all) ε, $0 < \varepsilon < |I|$, then for all $\alpha \in \mathbb{R} \setminus \{0\}$ operator A_α has empty singular spectrum on $\operatorname{clos} I$.

Proof: Absence of singular spectrum

- $T_{\mu_\alpha} := -(T_\mu)^* : L^2(\mu_\alpha) \to L^2(\mu)$ SIO with kernel $(s-t)^{-1}$
- Regularization operators

$$(T_{\mu_\alpha})_\varepsilon : L^2(\mu_\alpha) \to L^2(w) : f \mapsto \int \frac{f(s)}{s-t+i\varepsilon}\, d\mu_\alpha(s)$$

  are uniformly bounded and $K f \mu_\alpha = w - \lim_{\varepsilon\to 0}(T_{\mu_\alpha})_\varepsilon f$
- Application to $f = \chi_{\operatorname{clos} I}$ yields with $d\nu := \chi_{\operatorname{clos} I} d\mu_\alpha$

$$\int_I |K\nu|^2 w(x) dx \leq C \|\chi_{\operatorname{clos} I}\|^2_{\mu_\alpha} < \infty$$

- Goluzina $\quad t\chi_{(\{|K\eta|>t\}\cap I)}\, dx \;\to\; 2\chi_I\, d|\eta_{\rm s}| + \chi_{\partial I}\, d|\eta_{\rm s}|$ $\square$

Summary

- Rank one perturbations give rise to certain SIO
- Rigidity
- Uniformly bounded regularizations of SIO for Borel measures
- Absence of singular spectrum for perturbed operators

Questions

Questions?

Proof: Regularizations (for T_ε)

- For $a \in \mathbb{R}$ and f, g with separated compact supports we have

$$\left| \iint f(t)\overline{g(s)} \, \frac{1 - e^{ia(s-t)}}{s-t} \, d\mu(t) d\nu(s) \right| \leq 2C \|f\|_\mu \|g\|_\nu$$

- $\varepsilon \int_0^\infty \frac{1-e^{ia(s-t)}}{s-t} \, e^{-\varepsilon a} da = \frac{1}{s-t+i\varepsilon}$ and $\varepsilon \int_0^\infty e^{-\varepsilon a} \, da = 1$, $\varepsilon > 0$
- By averaging the estimate over all $a \geq 0$ with weight $\varepsilon e^{-\varepsilon a}$

$$\left| \iint \frac{f(t)\overline{g(s)}}{s-t+i\varepsilon} \, d\mu(t) d\nu(s) \right| \leq 2C \|f\|_\mu \|g\|_\nu$$

for f, g with separated compact supports and uniformly in ε

We need to acquit ourselves of condition of separated supports!

Proof: Regularizations (for T_ε and general μ, ν)

Recall
$$f_n(t) := f_a(t)h_1(nt)\chi_{G_n}(t) + (1/2-\delta)f_s(t)\chi_{E_n}(t)$$
$$g_n(t) := g_a(t)h_2(nt)\chi_{G_n}(t) + (1/2-\delta)g_s(t)\chi_{F_n}(t)$$

- $f_n \to (1/2-\delta)f$ weakly in $L^2(\mu)$, because for any $k \in L^2(\mu)$
$$\int_{\mathbb{R}} \underbrace{f_a(t)h_1(nt)}\,\underbrace{\chi_{G_n}(t)\overline{k(t)}}\,d\mu(t) \to (1/2-\delta)(f_a,k)_\mu$$
- $\lim_{n\to\infty} \|f_n\|_\mu^2 = (\frac{1}{2}-\delta)\|f_a\|_\mu^2 + (\frac{1}{2}-\delta)^2\|f_s\|_\mu^2 \le (\frac{1}{2}-\delta)\|f\|_\mu^2$
- Similarly for g_n and g

Proof: Absence of singular spectrum

- Goluzina implies $|\{|K\nu| > t\} \cap I| \geq C/t > 0$ for $t \geq A$, if ν has a non-trivial singular part
- With this

$$\infty > \int_I |K\nu|^2 w(x)dx = \int_0^\infty 2t \int_{\{|K\nu|>t\}\cap I} w(x)\, dxdt$$
$$\geq \int_A^\infty 2t \int_{\{|K\nu|>t\}\cap I} w^*(x)\, dxdt \geq \int_A^\infty 2t \int_0^{C/t} w(x)dxdt$$
$$= \int_0^{C/A} [(C/x)^2 - A^2] w^*(x)dx$$

- Clearly $\int_0^{C/A} w^*(x)dx < \infty$