Singular Integrals and Rank One Perturbations

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This talk is based on joint work with S. Treil.

Outline

Setting

- A self-adjoint operator on Hilbert space $\mathcal{H}$, some vector φ
- Here for simplicity A bounded and $\varphi \in \mathcal{H}$
- Consider family of rank one perturbations

$$A + \alpha(\,\cdot\,, \varphi)\varphi \text{ for } \alpha \in \mathbb{R}$$

- WLOG $\varphi \in \mathcal{H}$ cyclic for A, i.e. $\mathcal{H} = \overline{\operatorname{span}\{A^n\varphi : n \in \mathbb{N}\}}$
- Question: How stable is the embedded singular spectrum when we change α?

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Reformulation using the spectral theorem

- $\mu := \mu^\varphi$ denotes the unique spectral measure of A wrt φ, i.e. $(A^n\varphi, \varphi) = \int t^n d\mu(t)$ for $n \in \mathbb{N}$
- There exists a unitary operator $U : \mathcal{H} \to L^2(\mu)$ such that $UA = M_t U$ and $U\varphi = \mathbf{1}$. Notation:

$$A, \varphi \text{ on } \mathcal{H} \quad \overset{U}{\sim} \quad M_t, \mathbf{1} \text{ on } L^2(\mu)$$

- $A + \alpha(\,\cdot\,, \varphi)\varphi$ on $\mathcal{H}$ $\overset{U}{\sim}$ $A_\alpha := M_t + \alpha(\,\cdot\,, \mathbf{1})\mathbf{1}$ on $L^2(\mu)$

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Definition of V

- Recall $A_\alpha = M_t + \alpha(\,\cdot\,, \mathbf{1}_t)\mathbf{1}_t$ on $L^2(\mu)$
- For the perturbed operator

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  i.e. for some unitary operator $V = V_\alpha : L^2(\mu) \to L^2(\mu_\alpha)$ we have $M_s V = V A_\alpha$ and $V\mathbf{1}_t = \mathbf{1}_s$
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Statement

Theorem (Representation formula)

Under the all the above assumptions, the unitary operator $V : L^2(\mu) \to L^2(\mu_\alpha)$, such that $M_s V = V A_\alpha$ is given by

$$Vf(s) = f(s) - \alpha \int \frac{f(s) - f(t)}{s - t}\, d\mu(t)$$

for all compactly supported C^1 functions f.

Proof

Recall $V : L^2(\mu) \to L^2(\mu_\alpha)$, $M_s V = V A_\alpha$, $V\mathbf{1}_t = \mathbf{1}_s$ and $A_\alpha = M_t + \alpha(\,\cdot\,, \mathbf{1}_t)\mathbf{1}_t$. With this we have

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Inductively assume $Vt^{n-1} = M_s^{n-1} V\mathbf{1}_t - \alpha \int \sum_{k=0}^{n-2} t^k s^{n-k-2} d\mu(t).$

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Statement

Theorem (Rigidity theorem)

Suppose μ on $\mathbb{R}$ is not a single atom and satisfies $\int (1+|t|)^{-1}\, d\mu(t) < \infty$. Assume that

$$Tf(s) = f(s) - \alpha \int \frac{f(s)-f(t)}{s-t}\, d\mu(t) \quad \text{for} \quad f \in C_0^1$$

extends to a bounded operator $L^2(\mu) \to L^2(\nu)$ with $\operatorname{Ker} T = \{0\}$.

Then there exists a function h such that $1/h \in L^\infty(\nu)$ and such that $M_h T : L^2(\mu) \to L^2(\nu)$ is unitary.

Moreover the unitary operator $U = M_h T$ gives the spectral representation of the operator $A_\alpha := M_t + \alpha(\,\cdot\,, \mathbf{1}_t)\mathbf{1}_t$ in $L^2(\mu)$, namely $UA_\alpha = M_s U$.

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Idea of proof

- Recall from linear algebra: Similar (i.e. $A = S^{-1}BS$) hermitian matrices are unitary equivalent
- Here $M_sT = T[M_t + \alpha(\,\cdot\,, \mathbf{1}_t)\mathbf{1}_t]$
- $\operatorname{Ker} T = \{0\}$ implies $\operatorname{Ker} T^* = \{0\}$
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Singular integral operator V

- f and g have separated compact supports, i.e. both $\operatorname{supp} f$ and $\operatorname{supp} g$ are compact and $\operatorname{dist}(\operatorname{supp} f, \operatorname{supp} g) > 0$
- Bounded operator $T : L^2(\mu) \to L^2(\nu)$ is a SIO, if

$$(Tf, g)_{L^2(\nu)} = \int \int K(s,t) f(t) \overline{g(s)}\, d\mu(t)\, d\nu(s)$$

for all $f \in L^2(\mu)$, $g \in L^2(\nu)$ with separated compact supports

Lemma

Operator $V : L^2(\mu) \to L^2(\mu_\alpha)$ from the representation theorem is a SIO with kernel $K(s,t) = -\alpha(s-t)^{-1}$.

In particular, $T_\mu := \alpha^{-1} V : L^2(\mu) \to L^2(\mu_\alpha)$ is a SIO with kernel $(t-s)^{-1}$ and $\|T_\mu\|_{L^2(\mu) \to L^2(\mu_\alpha)} \le |\alpha|^{-1}$.

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- f and g have separated compact supports, i.e. both $\operatorname{supp} f$ and $\operatorname{supp} g$ are compact and $\operatorname{dist}(\operatorname{supp} f, \operatorname{supp} g) > 0$
- Bounded operator $T : L^2(\mu) \to L^2(\nu)$ is a SIO, if

$$(Tf, g)_{L^2(\nu)} = \int\int K(s,t) f(t) \overline{g(s)} \, d\mu(t) \, d\nu(s)$$

for all $f \in L^2(\mu)$, $g \in L^2(\nu)$ with separated compact supports

Lemma

Operator $V : L^2(\mu) \to L^2(\mu_\alpha)$ from the representation theorem is a SIO with kernel $K(s,t) = -\alpha(s-t)^{-1}$.

In particular, $T_\mu := \alpha^{-1} V : L^2(\mu) \to L^2(\mu_\alpha)$ is a SIO with kernel $(t-s)^{-1}$ and $\|T_\mu\|_{L^2(\mu) \to L^2(\mu_\alpha)} \le |\alpha|^{-1}$.

Introduction

- For $\varepsilon > 0$ consider regularized operators

$$T_\varepsilon f(s) = \int \frac{f(t)}{s-t+i\varepsilon}\, d\mu(t) \quad \text{and}$$
$$\widetilde{T}_\varepsilon f(s) = \int_{|t-s|>\varepsilon} \frac{f(t)}{s-t}\, d\mu(t)$$

- T_ε, $\widetilde{T}_\varepsilon$ well defined for compactly supported f
- If $\int (1+x^2)^{-1}\, d\mu(x) < \infty$, then the operators are well defined by the above formulas for all $f \in L^2(\mu)$

Statement

Theorem (Regularizations)

Let μ and ν be inner regular Borel measures such that their singular parts are mutually singular, i.e. $\mu_{\mathrm{s}} \perp \nu_{\mathrm{s}}$, and such that

$$\left| \iint \frac{f(t)\overline{g(s)}}{s-t}\, d\mu(t)\, d\nu(s) \right| \leq C \|f\|_{L^2(\mu)} \|g\|_{L^2(\nu)}$$

for all $f \in L^2(\mu)$ and $g \in L^2(\nu)$ with separated compact supports. Then operators $T_\varepsilon, \widetilde{T}_\varepsilon : L^2(\mu) \to L^2(\nu)$ are uniformly bounded.

This theorem holds true for all kernels $K(s,t)$ that satisfy $|K(s,t)(s-t)| \leq C_1$ and $0 < C_2 \leq \operatorname{Re} K(s,t)(s-t)$. Now $T_\varepsilon^K \cdot = \int \frac{K(s,t)(s-t)\,\cdot}{s-t+i\varepsilon}\, d\mu(t)$ and $\widetilde{T}_\varepsilon^K \cdot = \int_{|t-s|>\varepsilon} K(s,t) \cdot d\mu(t)$.

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Absence of singular spectrum

- Question: How stable is the embedded singular spectrum of $A_\alpha = M_t + \alpha(\,\cdot\,, \mathbf{1}_t)\mathbf{1}_t$ when we change $\alpha \in \mathbb{R}$?
- Consider local weak Lebesgue space $L^{1,\infty}_{\text{loc}}(I) := \{f : |\{x \in K : |f| > \varepsilon^{-1}\}| = \mathcal{O}(\varepsilon), \varepsilon \to 0, \text{ for all } K \subset\subset I\}$

Corollary (Absence of singular spectrum)

Let A and A_α be as described above. Let $d\mu = wdt + d\mu_{\text{s}}$ be the Lebesgue decomposition of A's spectral measure.

If for an open interval I we have $\frac{1}{w} \in L^{1,\infty}_{\text{loc}}(I)$, then for all $\alpha \in \mathbb{R} \setminus \{0\}$ operator $A_\alpha = M_t + \alpha(\,\cdot\,, \mathbf{1}_t)\mathbf{1}_t$ has empty singular spectrum on I.

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- Showed how rank one perturbations give rise to certain SIO
- Rigidity
- Uniformly bounded regularizations of SIO for Borel measures
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Proof: Absence of singular spectrum (slide 1)

- $T_{\mu_\alpha} := -(T_\mu)^* : L^2(\mu_\alpha) \to L^2(\mu)$ SIO with kernel $(s-t)^{-1}$ and $\|T_{\mu_\alpha}\|_{L^2(\mu_\alpha)\to L^2(\mu)} \le |\alpha|^{-1}$
- Fix $\alpha \in \mathbb{R}$, $I_0 \subsetneq I$ bounded open interval
- $\nu := (\mu_\alpha)_{\mathrm{s}}|_{I_0}$ and $w_0 := w|_{I_0}$
- Regularization operators

$$(T_\nu)_\varepsilon : L^2(\nu) \to L^2(w_0) : f \mapsto \int \frac{f(s)}{s-t+i\varepsilon}\, d\nu(s)$$

  are uniformly bounded
- With $K\nu(t) := \lim_{\varepsilon\to 0} \int \frac{d\nu(s)}{s-t+i\varepsilon}$ (exists dt a.e.) we have

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Proof: Absence of singular spectrum (slide 2)

- Assume $\nu \neq 0$
- Known $|\{|K\nu| > a\} \cap I_0| \geq C/a > 0$, large a
- Distribution function $d_w(a) := |\{w < a\} \cap I_0|$ on I_0
- Increasing rearrangement $w^* = d_w^{-1}$ of w on I_0
- So
$$\int_{\{|K\nu|>a\}\cap I_0} w\,dt \geq \int_0^{C/a} w^* dt \geq C/a^2 \qquad \text{large } a$$
- $1/w \in L^{1,\infty}_{\text{loc}}(I) \Rightarrow d_w(t) = |\{1/w > 1/t\} \cap I_0| \leq Ct$, small t
- Therefore $w^*(t) \geq t/C$ and
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