Regularizations of general singular integral operators

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This talk is based on joint work with Sergei Treil.
(Sergei's title of this talk: 'Singular integrals made easy')

Outline

The problem

- Consider kernel $K(s,t)$ which is singular near the diagonal, i.e. $K(s,\,\cdot\,)$ and $K(\,\cdot\,,t)$ are not in L^1_{loc} near $s=t$
- A SIO is an operator T on $L^2(\mu)$ is *formally* given by

$$Tf(s) = \int K(s,t)f(t)\,d\mu(t)$$

- Integral is not defined even for the simplest functions f
- What meaning do we give this formal expression?
- If T is the classical Hilbert Transform on the real line, then the integral exists in the sense of principal value

$$\lim_{\alpha\to 0+}\int_{|s-t|>\alpha}\frac{f(t)}{s-t}dt \qquad \text{for } f\in C^1_{\text{c}}(\mathbb{R})$$

- This approach is not possible, if $d\mu(t)\neq dt$

Some widely accepted remedies

- In the general situation the boundedness in L^p is often defined as the uniform boundedness (independent of $\varepsilon \to 0$) of either

truncated operators: $T_\varepsilon f(s) = \displaystyle\int_{|s-t|>\varepsilon} K(s,t) f(t) d\mu(t)$, or

smooth regularizations, e.g.: $T_\varepsilon f(s) = \displaystyle\int_{\mathbb{R}} \frac{f(t)}{s-t+i\varepsilon} d\mu(t)$

- The *corresponding SIO* is then the limit point (in WOT)
- T is unique up to '+multiplication by $L^\infty-$function'

Definitions

- Radon measures μ and ν in $\mathbb{R}^N$ (no grow, doubling condition)
- A *singular kernel* in $\mathbb{R}^N$ (wrt μ and ν) is locally $L^2(\mu \times \nu)$ off the diagonal $\{(s,t) \in \mathbb{R}^N \times \mathbb{R}^N : s = t\}$
- K is *of order* d, if the kernel $\widetilde{K}$ is locally $L^2(\mu \times \nu)$,

$$\widetilde{K}(s,t) = \begin{cases} K(s,t)|s-t|^d, & s \neq t \\ 0 & s = t \end{cases}$$

- Formal singular integral operator with the kernel K is *restrictedly bounded* in L^p if

$$\left| \iint K(s,t) f(t) g(s) d\mu(t) d\nu(s) \right| \leq C \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\nu)},$$

  for all bounded f, g with separated compact supports
- The least C (for p, μ and ν fixed) is the *restricted norm* of K

A trivial idea

- Recall (from previous slide): K is restrictedly bounded in L^p if

$$\left|\iint K(s,t)f(t)g(s)d\mu(t)d\nu(s)\right| \leq C\|f\|_{L^p(\mu)}\|g\|_{L^{p'}(\nu)},$$

  for all bounded f, g with separated compact supports
- If $K(s,t)$ has restricted L^p norm C, then the kernel

$$K(s,t)e^{-ia\cdot t}e^{ia\cdot s} \quad \text{for any } a \in \mathbb{R}^N$$

  is also restrictedly bounded with the same constant
- Averaging over all a with weight $\rho \in L^1(dx)$, kernel

$$\int_{\mathbb{R}^N} \rho(a)K(s,t)e^{-ia\cdot t}e^{ia\cdot s}da = \widehat{\rho}(t-s)K(s,t)$$

  is also restrictedly bounded with restricted norm $C\|\rho\|_1$

Lemma

Let K be a restrictedly L^p bounded kernel with a bound C. Assume that $\rho \in L^1(dx)$ and let $M = 1 - \widehat{\rho}$, $M_\varepsilon(x) := M(x/\varepsilon)$. Then the kernels $K_\varepsilon(s,t) := K(s,t)M_\varepsilon(t-s)$ are L^p restrictedly bounded with constant $(1 + \|\rho\|_1)C$.

- On $\mathbb{R}$ consider Hilbert Transform $K(s,t) = \pi^{-1}(s-t)^{-1}$ and the weight $\rho(x) = e^{-x}\chi_{[0,\infty)}(x)$
- We get $\widehat{\rho}(s) = (1+is)^{-1}$, so $M(s) = 1 - \widehat{\rho}(s) = \frac{s}{s-i}$ and

$$M_\varepsilon(s) := M(s/\varepsilon) = \frac{s}{s - i\varepsilon}$$

- Regularization with this mollifying factor gives the kernel

$$K_\varepsilon(s,t) = \frac{1}{\pi} \cdot \frac{1}{s-t} M_\varepsilon(t-s) = \frac{1}{\pi} \cdot \frac{1}{s-t+i\varepsilon}$$

Family of smooth mollifying (Schur) multipliers

Theorem

Let M be a function on $\mathbb{R}^N$ such that $|M(x)| \le C|x|^d$ near 0, and $\rho = 1 - M \in H^k(\mathbb{R}^N) = W^{k,2}(\mathbb{R}^N)$, $k > N/2$.

Then the functions $M_\varepsilon(s,t) := M((t-s)/\varepsilon)$ is a family of smooth regularizing multipliers, meaning that:

(i) *$M_\varepsilon(s,t) \to 1$ as $\varepsilon \to 0$ uniformly on all sets $\{s, t \in \mathbb{R}^N : |s-t| > a\}$, $a > 0$.*

(ii) *For any singular kernel K of order d the regularized kernels $K_\varepsilon = K M_\varepsilon$ are in $L^2_{\mathrm{loc}}(\mu \times \nu)$.*

(iii) *If the kernel is restrictedly bounded in L^p, and the measures μ and ν do not have common atoms, then the regularized integral operators T_ε with kernels K_ε are uniformly (in ε) bounded as operators $L^p(\mu) \to L^p(\nu)$.*

Main application: Uniform boundedness

- Let T be a limit point of T_ε, $\varepsilon \to 0$ in WOT
- As in the classical case, T is unique up to '+multiplication by $L^\infty-$function'

Theorem

Let μ and ν be Radon measures in $\mathbb{R}^N$ without common atoms. Assume that a singular kernel K is L^p resrictedly bounded, with the restricted norm C.

Then the integral operator with T kernel K is a bounded operator $L^p(\mu) \to L^p(\nu)$ with the norm at most $2(1 + \|\rho\|_1)C$.

Case of common atoms

When μ and ν do have common atoms, the boundedness of the singular integral operator can be defined as follows:

- Decompose
$$\mu = \widetilde{\mu} + \mu_0, \qquad \nu = \widetilde{\nu} + \nu_0,$$
where μ_0 and ν_0 are the parts of μ and ν supported on their common atoms
- Use the Theorem to check the L^p boundedness as an operator $L^p(\mu) \to L^p(\widetilde{\nu})$ or $L^p(\widetilde{\mu}) \to L^p(\nu)$
- It remains to check the block acting $L^p(\mu_0) \to L^p(\nu_0)$. But the bilinear form of this block is well defined for functions supported at finitely many points, so there is no problem defining this block

Idea of Theorem

- Under some additional assumptions, the restricted L^p boundedness implies the uniform boundedness of the truncated operators T_ε,

$$T_\varepsilon f(s) = \int_{|s-t|>\varepsilon} K(s,t) f(t) d\mu(t)$$

- These assumptions are satisfied for classical operators like generalized Riesz Transforms (treated as a vector-valued transformation), or Ahlfors–Beurling operator

Summary

What is a singular integral operator (SIO)?

Smooth mollifying multipliers

- A trivial idea
- Schur multipliers
- Uniform boundedness of regularized kernels

Uniform boundedness of truncations

This work is accepted by 'Revista Matematica Iberoamericano' for publication and available on the arXiv:1010.6184. For further questions please do not hesitate to contact me.

Thank you for your attention.