A geometric approach to rank one perturbations via the theory of dilations

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This talk is based on joint work with Ronald G. Douglas.

Spectral theory of perturbations

- Consider self-adjoint operators A and B on a separable Hilbert space
- Question: Given the spectral measure μ_A, what can be said about μ_{A+B}, if $B \in$ Class X?
- Here Class $X = \{$rank one$\}$
- L.: In a certain sense, rank one perturbations are as difficult as random Hamiltonians!

Aleksandrov–Clark theory

- Take $\theta \in H^\infty(\mathbb{D})$ with $\theta(0) = 0$
- Define $\triangle = \sqrt{1-|\theta|^2}$ and $\bar{\triangle} = \begin{cases} 0, & \text{if } \sqrt{1-|\theta|^2} = 0, \\ 1, & \text{if } \sqrt{1-|\theta|^2} > 0 \end{cases}$
- Sz.-Nagy–Foias model space $K_\theta = \begin{pmatrix} H^2 \\ \bar{\triangle} L^2 \end{pmatrix} \ominus \begin{pmatrix} \theta \\ \triangle \end{pmatrix} H^2$
- All unitary rank one perturbations of $S_\theta = P_{K_\theta} M_z|_{K_\theta}$ are

$$\widetilde{U}_\gamma = S_\theta + \gamma \left(\,\cdot\,, \begin{pmatrix} \bar{z}\theta \\ \bar{z}\triangle \end{pmatrix} \right)_{K_\theta} \begin{pmatrix} \mathbf{1}_z \\ 0 \end{pmatrix} \qquad \text{on } K_\theta \text{ for } |\gamma| = 1$$

- Let μ_γ denote the spectral measure of $\widetilde{U}_\gamma$
- (θ inner) $\Leftrightarrow$ ($K_\theta = H^2 \ominus \theta H^2$) $\Leftrightarrow$ (μ_γ purely singular)

Normalized Cauchy transform

- Let $f \in L^2(\mu_\gamma)$
- Cauchy transform

$$K(f\mu_\gamma)(z) = \int \frac{f(\xi)d\mu_\gamma(\xi)}{1-\bar{\xi}z}$$

- Via Herglotz argument $K\mu_\gamma(z) = (1-\bar{\gamma}\theta(z))^{-1}$ for $z \in \mathbb{C}\backslash\mathbb{T}$
- Normalized Cauchy transform

$$\mathcal{C}_{f\mu_\gamma} = \frac{K(f\mu_\gamma)}{K\mu_\gamma}$$

- For inner θ we have $\mathcal{C}_{f\mu_\gamma} \in K_\theta$
- Poltoratski: The non-tangential boundary limit obeys

$$\lim_{\mathbb{D}\ni z\to\zeta} \mathcal{C}_{f\mu_\gamma}(z) = f(\zeta) \qquad (\mu_\gamma)_{\mathrm{s}} - \text{a.e. } \zeta \in \mathbb{T}$$

Definitions

- Let μ probability measure on $\mathbb{T}$
- Consider completely nonunitary (cnu) contraction which has defect spaces $\mathcal{D}_{U_0} =< \bar{\xi} >$ and $\mathcal{D}_{U_0^*} =< \mathbf{1}_\xi >$:

$$U_0 = M_\xi - (\,\cdot\,, \bar{\xi})_{L^2(\mu)} \mathbf{1}_\xi \quad \text{on } L^2(\mu)$$

- Clark 1972: All unitary rank one perturbations of U_0 given by

$$U_\gamma = U_0 + \gamma(\,\cdot\,, \bar{\xi})_{L^2(\mu)} \mathbf{1}_\xi, \qquad |\gamma| = 1$$

- For $|\gamma| < 1$ the family $\{U_\gamma\}$ consists of cnu contractions

Characteristic function of U_0

- $\vartheta \in H^\infty(\mathbb{D})$ characteristic function of U_0 such that

$$\Theta(z).\bar{\xi} = \vartheta(z)\mathbf{1}_z = \vartheta(z)$$

- U_0 is unitary equivalent to $S_\vartheta = P_{K_\vartheta} M_z|_{K_\vartheta}$ on the model space K_ϑ

Theorem (D.–L.)

The characteristic function of the cnu contraction U_0 is given by

$$\vartheta(z) = z\mathcal{C}_{\bar{\xi}\mu} = \gamma z \mathcal{C}_{\bar{\xi}\mu_\gamma} \qquad \textit{for } z \in \mathbb{D}, |\gamma| = 1.$$

In particular, we have $\mathcal{C}_{\bar{\xi}\mu} \in H^\infty(\mathbb{D})$.

- Abakumov–L.–Poltoratski: $\mathcal{C}_{f\mu} = \mathcal{C}_{f_\gamma \mu_\gamma}$, $(\bar{\xi})_\gamma = \gamma\bar{\xi}$

Identification of characteristic functions

- Mitkovski(–Foiaş): Identification $\Theta(z).\bar{\xi} = \theta(z)\mathbf{1}_z = \theta(z)$
- Therefore we have

$$\vartheta(z) \equiv \theta(z)$$

Corollary (Special case of Poltoratski's theorem)

For every $\gamma \in \mathbb{T}$ *the non-tangential limit*

$$\lim_{z\to\zeta} \mathcal{C}_{\bar{\xi}\mu_\gamma}(z) \to \bar{\zeta} \qquad (\mu_\gamma)_{\mathrm{s}} - \textit{a.e. } \zeta \in \mathbb{T}.$$

Corollary

For every $\gamma \in \mathbb{T}$ *the non-tangential boundary limit*

$$\lim_{z\to\zeta} \vartheta(z) = \gamma \qquad (\mu_\gamma)_{\mathrm{s}} - \textit{a.e. } \zeta \in \mathbb{T}.$$

Radial jump behavior of $\mathcal{C}_{\bar{\xi}\mu_\gamma}$

- Consider the radial jump of the normalized Cauchy transform at $\zeta \in \mathbb{T}$:

$$[[\mathcal{C}_{f\mu}]](\zeta) = \lim_{r\to 1} \left(\mathcal{C}_{f\mu}(r\zeta) - \mathcal{C}_{f\mu}(\zeta/r)\right)$$

- Further we use the notation $K^{\pm}\mu(z) = \lim_{r\to 1^{\pm}} K\mu(rz)$

Theorem (D.–L.)

Radial jump behavior of $\mathcal{C}_{\bar{\xi}\mu}$ *across the unit circle is given by*

$$[[\mathcal{C}_{\bar{\xi}\mu}]](z) = \frac{1}{z(K^+\mu(z))(K^-\mu(z))} \frac{d\mu}{dm}(z) \qquad \text{for } m - \text{a.e. } z \in \mathbb{T}.$$

Dilation of U_0

- $U_0 = M_\xi - (\,\cdot\,, \bar{\xi})_{L^2(\mu)} \mathbf{1}_\xi$ is a cnu contraction on $L^2(\mu)$
- The minimal unitary (power) dilation W_γ 'lives' on

$$\mathcal{K} = \overline{H^2_{<w>}} \oplus L^2(\mu) \oplus H^2_{<\mathbf{1}_w>}$$

- W_γ initial space decomposition

$$\mathcal{K} = \bar{w}\overline{H^2_{<w>}} \oplus < \bar{w} > \oplus < \bar{\xi} >^\perp \oplus < \bar{\xi} > \oplus H^2_{<\mathbf{1}_w>}$$

- W_γ range space decomposition

$$\mathcal{K} = \overline{H^2_{<w>}} \oplus < \mathbf{1}_\xi >^\perp \oplus < \mathbf{1}_\xi > \oplus < \mathbf{1}_w > \oplus w H^2_{<\mathbf{1}_w>}$$

Matrix representation of W_γ

- Take $\beta = (1 - |\gamma|^2)^{1/2}$
- Note $|\gamma| = 1$ implies $\beta = 0$

With these decompositions, the matrix representation of the dilation W_γ is

$$\underbrace{\begin{array}{c} \overline{H^2_{<w>}} \\ <\mathbf{1}_\xi>^\perp \\ <\mathbf{1}_\xi> \\ <\mathbf{1}_w> \\ wH^2_{<\mathbf{1}_w>} \end{array}}_{\text{range decomp.}} \leftarrow \underbrace{\left(\begin{array}{cc|cc|c} \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} & P_{<\mathbf{1}_\xi>^\perp} U_0|_{<\bar{\xi}>^\perp} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \gamma & \mathbf{O} & \beta & \mathbf{O} \\ \hline \mathbf{O} & \beta & \mathbf{O} & \gamma & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} \end{array}\right)}_{=[W_\gamma]} \leftarrow \underbrace{\begin{array}{c} \bar{w}\overline{H^2_{<w>}} \\ <\bar{w}> \\ <\bar{\xi}>^\perp \\ <\bar{\xi}> \\ H^2_{<\mathbf{1}_w>} \end{array}}_{\text{initial sp. decomp.}}$$

Rank one perturbation of W_γ

Consider the rank one perturbation

$$\widetilde{W}_\gamma = W_\gamma + < \cdot, \bar{w} \oplus -\bar{\xi} \oplus 0 > (0 \oplus (\beta-\gamma)\mathbf{1}_\xi \oplus (\gamma-\beta)\mathbf{1}_w).$$

The matrix representations of W_γ and $\widetilde{W}_\gamma$ wrt the bases above are

$$\begin{array}{c} \overline{H^2_{<w>}} \\ <\mathbf{1}_\xi>^\perp \\ <\mathbf{1}_\xi> \\ <\mathbf{1}_w> \\ wH^2_{<\mathbf{1}_w>} \end{array} \leftarrow \left(\begin{array}{cc|cc|c} \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} & P_{<\mathbf{1}_\xi>^\perp} U_0|_{<\bar{\xi}>^\perp} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \gamma & \mathbf{O} & \beta & \mathbf{O} \\ \hline \mathbf{O} & \beta & \mathbf{O} & \gamma & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} \end{array}\right) \leftarrow \begin{array}{c} \bar{w}\overline{H^2_{<w>}} \\ <\bar{w}> \\ <\bar{\xi}>^\perp \\ <\bar{\xi}> \\ H^2_{<\mathbf{1}_w>} \end{array}$$

$$\begin{array}{c} \overline{H^2_{<w>}} \\ <\mathbf{1}_\xi>^\perp \\ <\mathbf{1}_\xi> \\ <\mathbf{1}_w> \\ wH^2_{<\mathbf{1}_w>} \end{array} \leftarrow \left(\begin{array}{cc|cc|c} \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} & P_{<\mathbf{1}_\xi>^\perp} U_0|_{<\bar{\xi}>^\perp} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \beta & \mathbf{O} & \gamma & \mathbf{O} \\ \hline \mathbf{O} & \gamma & \mathbf{O} & \beta & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} \end{array}\right) \leftarrow \begin{array}{c} \bar{w}\overline{H^2_{<w>}} \\ <\bar{w}> \\ <\bar{\xi}>^\perp \\ <\bar{\xi}> \\ H^2_{<\mathbf{1}_w>} \end{array}$$

$\widetilde{W}_1 \cong$Bilateral shift $\oplus\, U_\gamma$

$$\begin{array}{c} \overline{H^2_{<w>}} \\ <\mathbf{1}_\xi>^\perp \\ <\mathbf{1}_\xi> \\ <\mathbf{1}_w> \\ wH^2_{<\mathbf{1}_w>} \end{array} \leftarrow \underbrace{\left(\begin{array}{cc|cc|c} \mathbf{I} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} & P_{<\mathbf{1}_\xi>^\perp} U_0|_{<\bar{\xi}>^\perp} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \beta & \mathbf{O} & \gamma & \mathbf{O} \\ \hline \mathbf{O} & \gamma & \mathbf{O} & \beta & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{I} \end{array}\right)} \leftarrow \begin{array}{c} \bar{w}\overline{H^2_{<w>}} \\ <\bar{w}> \\ <\bar{\xi}>^\perp \\ <\bar{\xi}> \\ H^2_{<\mathbf{1}_w>} \end{array}$$

$$= \left[\widetilde{W}_\gamma\right]$$

- For $\gamma = 1$ we have $U_1 = M_\xi = P_{L^2(\mu)} \widetilde{W}_1|_{L^2(\mu)}$
- Further $\widetilde{W}_1 \cong$ (bilateral shift on $L^2(\mathbb{T})$) $\oplus$ (M_ξ on $L^2(\mu)$)

Location of absolutely continuous spectrum

Recall that $\bar{\triangle}(z) = \begin{cases} 0, & \text{if } \sqrt{1-|\vartheta|^2} = 0, \\ 1, & \text{if } \sqrt{1-|\vartheta|^2} > 0. \end{cases}$

Theorem

Measure $\bar{\triangle} dm$ is mutually equivalent to the absolutely continuous part $d(\mu_\gamma)_{\mathrm{ac}}$ for all $|\gamma| = 1$.

Idea of proof.

- Schreiber, Sz.-Nagy–Foias: W_γ is purely absolutely continuous
- Spectral mult. of minimal unitary dilation equals $1 + \bar{\triangle}(z)$
- $\widetilde{W}_1 \cong$ (biliateral shift on $L^2(\mathbb{T})$) $\oplus$ (U_1 on $L^2(\mu)$)
- Kato–Rosenblum

Characteristic functions of U_γ, $|\gamma| < 1$

Recall $U_\gamma = U_0 + \gamma < \cdot, \bar{\xi} > \mathbf{1}_\xi$, $|\gamma| < 1$ cnu contractions.

Theorem (D.–L.)

The characteristic functions of U_γ, $|\gamma| < 1$ are

$$\vartheta_\gamma(z) = -\gamma + z(1 - |\gamma|^2)\frac{K(\bar{\xi}\mu)}{K\mu - \bar{\gamma} z K(\bar{\xi}\mu)}, \qquad |\gamma| < 1.$$

Corollary (Livsic)

The characteristic function ϑ_γ is related to $\vartheta = \vartheta_0$ via a linear fractional transformation

$$\vartheta_\gamma(z) = \frac{-\gamma + \vartheta(z)}{1 - \bar{\gamma}\vartheta(z)} \qquad |\gamma| < 1.$$

Summary and Outlook

Summary

We studied unitary rank one perturbations via the alternative, geometric, framework of dilation theory. Within this framework, we were able to embed several other approaches from literature, and deduced several new results.

1. Characteristic function of U_0
2. Dilation Theory
3. Characteristic function of U_γ, $|\gamma| < 1$

Outlook

- Higher, but equal deficiency indices
- Interpret the relationship of ϑ_γ, $|\gamma| < 1$, and ϑ
- Jump behavior of the normalized Cauchy transform for more general functions, and relation to other subjects.